Arithmetic of metaplectic modular forms

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A Thesis presented for the degree of
Doctor of Philosophy

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United Kingdom

June 2019

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Abstract: Modular forms came to the attention of number theorists through the wealth of their arithmetic behaviour, the development and applications of which continue to surprise. Arithmetic data of associated L-functions have conjectured links to fundamental questions, for example the generalised Riemann hypothesis and the BSD conjecture; special values of L-functions and their p-adic analogues have had a key role in progress towards BSD. Modular forms of half-integral weight have a number-theoretic history spanning as far back as that of their integral-weight counterparts, but their arithmetic theory has long been latent. Being fundamental variants of integral-weight modular forms, a fully fledged theory of half-integral weight modular forms has high potential for impact in areas of number theory.

In this thesis, we develop four key areas in the arithmeticity of Siegel modular forms of half-integral weight, focusing on the behaviour of their Fourier coefficients and associated L-functions as follows: an analogue of Garrett's conjecture on the precise algebraicity of Klingen Eisenstein series and of the decomposition $\mathcal{M}_k = \mathcal{S}_k \oplus \mathcal{E}_k$; the precise algebraicity of special L-values; the existence of p-adic L-functions; and, for vector-valued modular forms, an explicit Rankin-Selberg integral expression. Some of the results, such as special values of L-functions, are further refinements of existing theorems; others, such as the construction of p-adic L-functions, are entirely new.

The multifaceted nature of modular forms is a considerable characteristic of theirs. Classically developed as analytic objects, integral-weight modular forms have been reinterpreted algebraically in terms of automorphic representations and associations to motives. Since the algebraic viewpoint remains insufficient for our purposes we focus on the analytic theory and methods of proof for half-integral weight modular forms, using Shimura's theory of Hecke operators and his Rankin-Selberg expression as a basis, and modifying the established methods of Harris, Sturm, and Panchishkin to prove our results.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification.

Sections 4.2.2 and 4.7 are contained in a paper that has been accepted for publication in *manuscripta mathematica*, [Mer18a]. Section 2.3 and Chapter 3 are contained in the preprint [Mer18b], and Chapter 4 can also be found in the preprint [Mer19]. The work in Chapter 5 is based on joint research with Thanasis Bouganis, Durham University, and is contained in [BM18]. All preprints are under review.

Acknowledgements

I would like to thank my supervisor Thanasis Bouganis, whose enthusiasm and guidance encouraged this work beginning to end and from whom I have learnt a huge amount. I am especially grateful for the time we spent collaborating, which resulted in Chapter 5 of this thesis. I would also like to thank my examiners Herbert Gangl and Alexei Panchishkin for their time and helpful comments.

The Department of Mathematical Sciences of Durham University formed a great place to work and I am grateful to the academic staff and fellow PhD students for fostering this with regular seminars and events, as well as the hard work of the office and cleaning staff who keep it all together.

I would like to acknowledge various people from in and out of Durham who have helped over the past years in both academic and non-academic capacities: officemates, conference buddies from the department, and family and friends from Durham, home, and abroad. Final thanks go to my parents for their boundless support.

This work was supported by an Engineering and Physical Sciences Research Council PhD scholarship (Grant No. 000118421).

Contents

Chapter 1

Introduction

One of the first occurrences of modular forms was of half-integral weight: Gauss had discovered the classical theta series

$$P(t) := 1 + 2\sum_{n\in\mathbb{Z}} e^{-n^2\pi t}, \qquad \mathfrak{Re}(t) > 0,$$

at least by 1808, [Roy17, p. 32]. This series arose from the heat diffusion equation on the real line and it satisfies, by the Poisson summation formula, a half-integral weight transformation of the form

$$P(t) = \frac{1}{\sqrt{t}} P\left(\frac{1}{t}\right),$$

and in fact putting $\theta(z) := P(-iz)$ does give a well-known modular form of weight $\frac{1}{2}$ – the theta series. Such series have surprisingly direct arithmetic applications, for example taking the rth power gives

$$\theta(z)^r = \sum_{n=0}^{\infty} s_r(n) e^{2\pi i n z}, \tag{1.0.1}$$

where $s_r(n)$ is the number of ways that n can be represented as the sum of r squares. For $r = 4$ this can be used to prove that any positive integer is the sum of four squares along with the explicit value

$$s_4(n) = \sum_{4\nmid d|n} d.$$

Since θ^r is a modular form of weight $r/2$, the purely arithmetic question of how many ways a positive integer can be expressed as the sum of an odd number of squares is directly related to the theory of half-integral weight modular forms, and this constitutes one of the earliest instances of the notoriously fecund relationship between number theory and modular forms. So it is more than a little surprising that the analytic theory of classical half-integral weight modular forms remained inchoate until Shimura's paper in 1973, [Shi73].

It took even longer for Weil's theory of theta series and the metaplectic group in [Wei64] to filter through and be used to establish the analytic theory of half-integral weight Siegel modular forms, which we dub *metaplectic modular forms.* Key aspects of this theory, such as the notion of Hecke operators and the standard L-functions, were established by Shimura in [Shi95b]. So whilst the arithmetic theory of integral-weight Siegel modular forms developed significantly in, amongst others, the work of Harris [Har81], Sturm [Stu81], Panchishkin [Pan91], and Böcherer and Schmidt [BS00], that of metaplectic modular forms lagged behind. Recently, however, this has been an active research area, see for example algebraic L-values being determined by Shimura [Shi00] and Bouganis [Bou18], as well as the analytic theory of p-adic metaplectic forms being developed by Ramsey in [Ram06].

That automorphic forms have algebraic interpretations was a key development in the latter half of the last century. This involves viewing automorphic forms as automorphic representations and studying Galois representations of a motive that one can associate to the automorphic form. The most famous example of this link between motives and automorphic forms is Wiles' modularity theorem of [Wil95], which gave the association of classical weight-two modular forms and elliptic curves. The general philosophy of the Langlands programme asserts that such associations exist and that the L-functions one constructs on either side are equivalent. Concerns on the arithmetic properties of motives, which often arise out of fundamental problems in number theory, can thus be illuminated by looking at the corresponding arithmetic questions on the associated automorphic forms, should this association be known. The algebraic theory of metaplectic modular forms is much less understood than the analytic theory, though serious progress is being made in this regard, most notably by Weissman in [Weiss18], as well as McNamara [McNam12], Gan and Gao [GG18, Gao18], in developing L-groups for general metaplectic covers (including Mp_n and those of GL_n). The key motif of this thesis is the capitalisation on the lopsided nature, with respect to the analytic-algebraic dichotomy, of the theory of metaplectic modular forms in order to establish key arithmetic properties via analytic means.

The notion of arithmeticity in the theory of automorphic forms is vague and encompasses many aspects. Two key facets that form the focus of our investigations are **(1)** modular forms with algebraic Fourier coefficients and **(2)** arithmetic properties of L-functions associated to eigenforms. We have already encountered an example of **(1)** as the theta series θ^r, defined in (1.0.1), which has coefficients in $\mathbb{Z}$. The not-at-all obvious fact that forms with algebraic coefficients span spaces of modular forms underlines the importance of such arithmetic forms, for Siegel modular forms of integral and half-integral weight see [Shi00, Theorem 10.4 (3)] and [Shi00, Theorem 10.7 (3)] respectively. Moreover, L-functions associated to eigenforms encode further arithmetic data and we explore **(2)** more thoroughly in the following paragraph. Before doing so, however, we note that **(1)** and **(2)** are not necessarily mutually exclusive areas and the previously mentioned work of Shimura, [Shi73], and the subsequent investigations of Waldspurger, [Wal81], give a nice example of interplay between them. The salient result of [Shi73] is Shimura's correspondence which goes between classical modular forms of half-integral weight and those of integral weight, with standard L-functions of the former being quadratic twists of those of the latter. In [Wal81], Waldspurger showed that Fourier coefficients of half-integral weight modular forms can be expressed as an algebraic special value of the standard L-function associated to the corresponding integral-weight form.

The history of L-functions and their arithmetic properties predates even that of modular forms and can be traced back to at least 1735 with Euler's solution to the Basel problem, which sought the value of

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

or, in other words, the special value at $s = 2$ of the proto-L-function $\zeta(s)$ – the Riemann zeta function. Euler calculated this to be $\frac{\pi^2}{6}$ and that we have $\pi^{-2}\zeta(2) \in \mathbb{Q}$ is a deep property with analogues for more general L-functions. The general importance of such functions is clear from their key roles in long-standing unsolved conjectures such as the classical and generalised Riemann hypothesis, and the Birch and Swinnerton-Dyer (BSD) conjecture. The latter conjectures that the rank of an elliptic curve E is precisely the order of the zero of its L-function $L(E, s)$ at $s = 1$ and a subsequent, stronger, statement of it predicts the value of the residue of $L(E, 1)$. In particular, for elliptic curves of rank zero, this stronger statement is the special value

$$\frac{L(E, 1)}{\Omega(E)} \in \mathbb{Q},$$

where $\Omega(E) \in \mathbb{R}$ is the period of E, and it moreover conjectures a precise value for this rational number in terms of certain invariants of the elliptic curve. The BSD conjecture therefore constitutes a surprising and direct connection between the arithmetic of L-functions and fundamental questions in number theory – in this case rational solutions to cubic equations. The focus of this thesis in this regard is on special values of the standard L-function associated to a metaplectic eigenform and is once again twofold: **(i)** the determination of the precise field extension of $\mathbb{Q}$ in which these special values belong; **(ii)** analytic interpolation of these special values to construct the p-adic L-function. In settings of automorphic forms for which the Langlands-theoretic correspondence is coherent, for example integral-weight Siegel modular forms, the concerns of both **(i)** and **(ii)** have deeper contextual interpretations. In the former, for general pure motives M, Deligne's conjecture of [Del79, Conjecture 2.8] for so-called "critical" values and Beilinson's conjecture [Bei85, Conjecture 3.4] for non-critical values of the L-function, $L(M, s)$, associated to M predict the existence of a number $c^+(M)$, known as the *period* of M, through which

$$\frac{L(M, s)}{c^+(M)} \in \overline{\mathbb{Q}}.$$

In [Bei85], Beilinson introduced "higher regulator" maps between motivic cohomology groups, which generalise the Dirichlet regulator map; in his conjecture the value of this quotient is given a precise prediction in terms of determinants of these regulator maps. In both conjectures, the value of $c^+(M)$ is also given an exact value. For **(ii)**, traditional interpretations of p-adic L-functions were analytic in nature, with the prototypical example being Kubota-Leopoldt's p-adic interpolation in [KL64] of the Riemann zeta function by constructing p-adic measures. However, such objects can be viewed in an *a priori* distinct and algebraic way. For example, in numerous papers [Iwa59, Iwa64, Iwa69], Iwasawa developed the theory of the Iwasawa algebra; it was his key insight to conjecture that the Kubota-Leopoldt p-adic zeta function is the generator of a certain characteristic ideal in the Iwasawa algebra of a Galois group – this is the original Iwasawa main conjecture, first proven by Mazur and Wiles in [MW84]. Provided that one has the necessary algebraic theory needed to construct the Iwasawa-theoretic objects, this conjecture can be generalised to other settings. Over GL_1 we can consider the p-adic interpolation of Dirichlet L-functions; in the GL_2 case, the Iwasawa main conjectures for classical modular forms of integral weight is a recent and active research area, see the work of Skinner and Skinner-Urban, [Ski06] and [SU14], and higher-dimensional cases are also being considered, for example by Wan in [Wan15]. The algebraic context of both **(i)** and **(ii)** are key to the general philosophy of Iwasawa theory, and the importance of this theory is given by its key role in the significant progresses made on the BSD conjecture. Naturally, all of these algebraic notions are presently unavailable in the setting of this thesis however we expect the analytic work here to eventually fit into the larger framework described above.

In Chapter 2, after recapping relevant foundational theory established by Shimura in [Shi93] and [Shi95b], we establish in Section 2.3 a new result on the algebraicity of Fourier coefficients of metaplectic Eisenstein series and give a subsequent algebraic decomposition of the space of modular forms into cusp forms and Eisenstein series. Given its eminent role in the methods of this thesis, we also give an exposition on the Rankin-Selberg method at the end of this chapter. Chapter 3 is dedicated to the algebraicity of special values of the standard L-function in this setting and the existence of the p-adic L-function is given in Chapter 4. Chapter 5 consists of joint work with Bouganis, in it the focus is shifted slightly to vector-valued modular forms, encompassing both integral and half-integral weight modular forms, and we extend the Rankin-Selberg method to this setting. As a result, analytic properties of the standard L-function are proved.

At the start of each section there will be a paragraph with notation, along with key assumptions, that have been introduced earlier on in the thesis.

Notation. Throughout the entire thesis, we fix a positive integer $1 \leq n \in \mathbb{Z}$.

Chapter 2

Metaplectic modular forms

The brute replacement of an integer weight k by a half-integer in the definition of a modular form causes the thorny issue of a consistent choice of square roots in the factor of automorphy; this issue is the genesis of the nuances and difficulties in the theory of metaplectic modular forms. The moniker of metaplectic modular forms is a result of this – one uses the metaplectic cover of Sp_n to make a consistent choice of roots. The first two sections of this chapter explore these nuances as we give the well-known foundational theory of metaplectic modular forms. The fundamental objects of study – L-functions associated to eigenforms – will be introduced at the end of Section 2.2. The first result of interest of this thesis is given in Section 2.3 in which we prove an algebraicity result regarding the well-known decomposition of modular forms into cusp forms and Eisenstein series. This result will be of direct use in the next chapter. To finish this chapter we give an expository section on the Rankin-Selberg method in Section 2.4, whose role within this thesis is crucial. General background and philosophy of the method is given as well as the derivation of the relevant integral expression involved.

2.1 Fourier expansions of modular forms

We begin by reviewing the definitions of modular forms, their adelisations, and we give the properties of their Fourier expansions. To do so, much of the notation and conventions used by Shimura in [Shi94, Shi95b, Shi00] are adopted and the material found here is adapted from these sources.

Let $\mathbb{A}_{\mathbb{Q}}$ and $\mathbb{I}_{\mathbb{Q}}$ denote the adele ring and idele group, respectively, of $\mathbb{Q}$; elements of $\mathbb{A}_{\mathbb{Q}}$ (resp. $\mathbb{I}_{\mathbb{Q}}$) have the form $(x_\infty, x_2, x_3, \dots, x_p, \dots)$, where $x_\infty \in \mathbb{R}$ (resp. $\mathbb{R}^\times$), $x_p \in \mathbb{Q}_p$ (resp. $\mathbb{Q}_p^\times$), and $x_p \in \mathbb{Z}_p$ (resp. $\mathbb{Z}_p^\times$) for all but finitely many primes p. The Archimedean place is denoted by ∞, the non-Archimedean places by $\mathbf{f}$, and if $x \in \mathbb{A}_{\mathbb{Q}}$ let $x_{\mathbf{f}} = (1, x_2, \dots, x_p, \dots)$. If G is any algebraic group, let $G_{\mathbb{A}}$ denote its adelisation (this is the group of all points of G with values in $\mathbb{A}_{\mathbb{Q}}$); let $G_\infty := G(\mathbb{R})$, $G_p := G(\mathbb{Q}_p)$, and $G_{\mathbf{f}}$ be the subgroup of $G_{\mathbb{A}}$ whose Archimedean place is the identity of G_∞. We view G as a subgroup of $G_{\mathbb{A}}$ by embedding diagonally at every place, but embed G_∞ and G_p into $G_{\mathbb{A}}$ place-wise. Specifically, for

$v \in \{\infty\} \cup \mathbf{f}$, this means viewing $g \in G_v$ as an element of $G_{\mathbb{A}}$ by defining $g_v = g$ and setting $g_{v'}$ as the identity of $G_{v'}$ for any $v \neq v' \in \{\infty\} \cup \mathbf{f}$.

For any fractional ideal $\mathfrak{r}$ of $\mathbb{Q}$ let $\mathfrak{r}_p$ denote the completion (with respect to the p-adic norm) of the localisation of $\mathfrak{r}$ at the prime p, which is an ideal of $\mathbb{Z}_p$, and understand $N(\mathfrak{r}) \in \mathbb{Q}_{\geq 0}$ to be the unique positive generator of $\mathfrak{r}$. For any element $t \in \mathbb{I}_{\mathbb{Q}}$ we denote by $t\mathfrak{r}$ the fractional ideal of $\mathbb{Q}$ such that $(t\mathfrak{r})_p = t_p\mathfrak{r}_p$ for any $p \in \mathbf{f}$. The adelic norm is

$$|x|_{\mathbb{A}} = |x_\infty| \prod_p |x_p|_p,$$

where $x \in \mathbb{A}_{\mathbb{Q}}$, $|\cdot| = |\cdot|_\infty$ denotes the usual absolute value on $\mathbb{R}$, and $|\cdot|_p$ denotes the p-adic absolute value for each $p \in \mathbf{f}$, normalised in the sense that $|p|_p = p^{-1}$. Any $x \in \mathbb{Q}_p$ has a p-adic expansion of the form $\pm \sum_{i \geq N} a_i p^i$, where $a_i \in \mathbb{Z}/p\mathbb{Z}$ and $a_N \neq 0$; the fractional part $\{x\} \in \mathbb{Q}$ of such an x is 0 if $N \geq 0$, otherwise $\{x\} := \pm \sum_{i=N}^{-1} a_i p^i$.

Define the unit circle by

$$\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\},$$

where $|z| := \sqrt{\mathfrak{Re}(z)^2 + \mathfrak{Im}(z)^2}$ for any $z \in \mathbb{C}$, and define three characters on $\mathbb{C}$, $\mathbb{Q}_p$, and $\mathbb{A}_{\mathbb{Q}}$ respectively, all with images in $\mathbb{T}$, by

$$\begin{aligned} e &: z \mapsto e^{2\pi i z}, \\ e_p &: x \mapsto e(-\{x\}), \\ e_{\mathbb{A}} &: x \mapsto e(x_\infty) \prod_{p \in \mathbf{f}} e_p(x_p). \end{aligned}$$

If $x \in \mathbb{A}_{\mathbb{Q}}$ and $z \in \mathbb{C}$ then we also put $e_{\mathbf{f}}(x) := e_{\mathbb{A}}(x_{\mathbf{f}})$, $e_\infty(x) = e(x_\infty)$, and $e_\infty(z) := e(z)$.

For any ring R and matrix $q \in M_n(R)$ we make use of the following notation: $q > 0$ (resp. $q \geq 0$) to mean that q is positive definite (resp. positive semi-definite), $\operatorname{tr}(q)$ to denote the sum of all diagonal elements of q,

$$\begin{aligned} |q| &:= \det(q), \\ \|q\| &:= |\det(q)|, \\ \tilde{q} &:= (q^T)^{-1}, \end{aligned}$$

with the last line defined only if q is invertible. If $R = \mathbb{A}_{\mathbb{Q}}$ we understand $\det(q)$ and $\operatorname{tr}(q)$ to be the element of $\mathbb{A}_{\mathbb{Q}}$ with $\det(q)_v = \det(q_v)$ and $\operatorname{tr}(q)_v = \operatorname{tr}(q_v)$, for $v \in \{\infty\} \cup \mathbf{f}$. For any collection $q_1, \ldots, q_\ell$ of matrices of arbitrary size with entries in R, let

$$\operatorname{diag}[q_1, \ldots, q_\ell] := \begin{pmatrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_\ell \end{pmatrix}$$

denote the matrix with q_j as its jth diagonal block and with zeros off the diagonal. If

$\alpha \in GL_{2n}(R)$ then put

$$\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix},$$

where $x_\alpha \in M_n(R)$ for $x \in \{a, b, c, d\}$.

For the rest of this thesis, fix

$$1 \leq n \in \mathbb{Z}.$$

Definition 2.1.1. We define an algebraic group G, subgroups $P, \Omega \leq G$, and the Siegel upper half-space $\mathbb{H}_n$ by

$$\begin{aligned} G :&= Sp_n(\mathbb{Q}) = \{\alpha \in GL_{2n}(\mathbb{Q}) \mid \alpha^T \iota \alpha = \iota\}, \qquad \iota := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \\ P :&= \{\alpha \in G \mid c_\alpha = 0\}, \\ \Omega :&= \{\omega \in G_\mathbb{A} \mid \det(c_\omega) \in \mathbb{I}_\mathbb{Q}\}, \\ \mathbb{H}_n :&= \{z = x + iy \in M_n(\mathbb{C}) \mid z^T = z, y > 0\}. \end{aligned}$$

A half-integral weight is an element $k \in \mathbb{Q}$ such that $k - \frac{1}{2} \in \mathbb{Z}$; an integral weight is an element $\ell \in \mathbb{Z}$. Recall in the classical $n = 1$ case that the factor of automorphy of weight $\ell \in \mathbb{Z}$ was given by $j(\gamma, z)^\ell = (c_\gamma z + d_\gamma)^\ell$, where $\gamma \in SL_2(\mathbb{Z})$. Since half-integral weights involve taking the square root of a complex number, we are confronted with the issue of a consistent choice of which root to take. This is resolved by making use of the metaplectic group – the double cover of the symplectic group. What actually happens here is that a certain subgroup of the metaplectic group gives roots of the factor of automorphy with *multiplier*, for instance, if $n = 1$ this is a root of $\pm(c_\gamma z + d_\gamma)$ whenever $\gamma \in SL_2(\mathbb{Z})$ is such that $b_\gamma \equiv c_\gamma \equiv 0 \pmod 2$.

In the classical construction of half-integral weight modular forms in [Shi73], Shimura used a theta series to construct a suitable factor of automorphy of weight $\frac{1}{2}$. Following the relatively recent exposé of the role of the symplectic group by Siegel in constructing a new type of higher-dimensional modular form, Weil in [Wei64] recast the Siegel theta series in a representation-theoretic light. From a certain unitary representation of the Heisenberg group one obtains a unitary representation, not of the symplectic group but of a central extension of it – the metaplectic group Mp_n. It is this representation of the metaplectic group that gives Weil's theory of theta series. Thus it is natural to see the emergent role of the metaplectic group in treating Siegel modular forms of half-integral weight. Indeed if one considers the case $n = 1$, where the Siegel and classical cases coincide, then the metaplectic group explicitly becomes the group $\mathfrak{G}$ used by Shimura in [Shi73] (we shall ourselves define this group later in Section 2.3.2). For general $n > 1$ it is difficult to give an analogously concrete description for Mp_n. Nevertheless their localizations $M_p := Mp_n(\mathbb{Q}_p)$, for any p, and the adelisation $M_\mathbb{A}$ of $Mp_n(\mathbb{Q})$ can be described as groups of unitary transformations on the spaces $L^2(\mathbb{Q}_p^n)$ (resp. $L^2(\mathbb{A}_\mathbb{Q}^n)$) of square-integrable functions $\mathbb{Q}_p^n \to \mathbb{C}$ (resp. $\mathbb{A}_\mathbb{Q}^n \to \mathbb{C}$), with the exact sequences

$$1 \to \mathbb{T} \to M_p \to G_p \to 1,$$

$$1 \to \mathbb{T} \to M_{\mathbb{A}} \to G_{\mathbb{A}} \to 1.$$

We have natural projections

$$\mathrm{pr}_{\mathbb{A}} : M_{\mathbb{A}} \to G_{\mathbb{A}},$$
$$\mathrm{pr}_p : M_p \to G_p,$$

either of which are denoted by pr when the context is clear. There are natural lifts

$$r : G \to M_{\mathbb{A}},$$
$$r_P : P_{\mathbb{A}} \to M_{\mathbb{A}},$$
$$r_\Omega : \Omega \to M_{\mathbb{A}},$$

through which we can and do view $G, P_{\mathbb{A}}$, and Ω as subgroups of $M_{\mathbb{A}}$. Moreover r_P and r_Ω are equal to r on P and $G \cap \Omega$ respectively, and satisfy

$$r_\Omega(\alpha\beta\gamma) = r_P(\alpha) r_\Omega(\beta) r_P(\gamma),$$

for $\alpha, \beta \in P_{\mathbb{A}}$ and $\gamma \in \Omega$ ([Shi95b, (1.2a, b)]).

For any two fractional ideals $\mathfrak{x}, \mathfrak{y}$ of $\mathbb{Q}$ such that $\mathfrak{x}\mathfrak{y} \subseteq \mathbb{Z}$, congruence subgroups are defined by the following respective subgroups of G_p, $G_{\mathbb{A}}$, and G:

$$\begin{aligned} D_p[\mathfrak{x}, \mathfrak{y}] &:= \{x \in G_p \mid a_x, d_x \in M_n(\mathbb{Z}), b_x \in M_n(\mathfrak{x}_p), c_x \in M_n(\mathfrak{y}_p)\}, \qquad p \in \mathbf{f}, \\ D[\mathfrak{x}, \mathfrak{y}] &:= Sp_n(\mathbb{R}) \prod_p D_p[\mathfrak{x}, \mathfrak{y}], \\ \Gamma[\mathfrak{x}, \mathfrak{y}] &:= G \cap D[\mathfrak{x}, \mathfrak{y}]. \end{aligned}$$

If $x, y \in \mathbb{Q}$ we shall often write $\Xi[x, y]$, for $\Xi \in \{D_p, D, \Gamma\}$, to mean $\Xi[x\mathbb{Z}, y\mathbb{Z}]$. The groups $D[\mathfrak{x}, \mathfrak{y}]$ can be thought of as "adelic" congruence subgroups, with their subgroup of global elements – $\Gamma[\mathfrak{x}, \mathfrak{y}]$ – giving the more familiar notion of congruence subgroups. Typically these will take the form $\Gamma[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$ for a fractional ideal $\mathfrak{b}$ and an integral ideal $\mathfrak{c}$ of $\mathbb{Q}$, and where we understand $\mathfrak{b}^{-1} = 0$ if $\mathfrak{b} = 0$. For modular forms of half-integral weight, there will be some restrictions on $\mathfrak{b}$ and $\mathfrak{c}$ which we explain next.

One of the fundamental differences in the theory of half-integral weight modular forms to that of integral-weight forms is in congruence subgroup restrictions. It is not the whole of the metaplectic group $M_{\mathbb{A}}$ that gives us roots of the factor of automorphy, but only a certain subgroup $\mathfrak{M} \leq M_{\mathbb{A}}$. Thus any congruence subgroup Γ we consider must be contained in $\mathfrak{M}$. In the classical case this meant half-integral weight forms had to have level divisible by 4 for example. In our setting we have that

$$\begin{aligned} C_p^\theta &:= \{\xi \in D_p[1,1] \mid (a_\xi b_\xi^T)_{ii} \in 2\mathbb{Z}_p, (c_\xi d_\xi^T)_{ii} \in 2\mathbb{Z}_p, 1 \leq i \leq n\}, \qquad p \in \mathbf{f}, \\ C^\theta &:= Sp_n(\mathbb{R}) \prod_p C_p^\theta, \\ \mathfrak{M} &:= \{\sigma \in M_{\mathbb{A}} \mid \mathrm{pr}(\sigma) \in P_{\mathbb{A}} C^\theta\}. \end{aligned}$$

Thus we must always ensure that we choose $\mathfrak{b}$ and $\mathfrak{c}$ such that $D[\mathfrak{b}^{-1}, \mathfrak{bc}] \leq \mathfrak{M}$. Since $D[2,2] \leq \mathfrak{M}$, the condition $\mathfrak{b}^{-1} \subseteq 2\mathbb{Z}$ and $\mathfrak{bc} \subseteq 2\mathbb{Z}$ ensures $D[\mathfrak{b}^{-1}, \mathfrak{bc}] \leq \mathfrak{M}$.

The above constitutes the groups and spaces with respect to which our modular forms are defined, and these spaces interact with each other as follows. The action of $Sp_n(\mathbb{R})$ on $\mathbb{H}_n$ is given by

$$\gamma \cdot z := (a_\gamma z + b_\gamma)(c_\gamma z + d_\gamma)^{-1},$$

for $\gamma \in Sp_n(\mathbb{R})$, $z \in \mathbb{H}_n$. That this is an action is given in [Kli90, pp. 2 – 3], in particular it is shown there that $c_\gamma z + d_\gamma \in GL_n(\mathbb{C})$. Further define

$$\begin{aligned} \Delta(z) &:= |\,\mathfrak{Im}(z)|, \\ \mu(\gamma, z) &:= c_\gamma z + d_\gamma, \\ j(\gamma, z) &:= |\mu(\gamma, z)|. \end{aligned}$$

If, now, $\alpha \in G_\mathbb{A}$ then $\alpha_\infty \in Sp_n(\mathbb{R})$ and define

$$\begin{aligned} \alpha \cdot z &:= \alpha_\infty \cdot z, \\ \mu(\alpha, z) &:= \mu(\alpha_\infty, z), \\ j(\alpha, z) &:= j(\alpha_\infty, z). \end{aligned}$$

If $\sigma \in M_\mathbb{A}$ with $\mathrm{pr}(\sigma) = \alpha \in G_\mathbb{A}$ then we set $x_\sigma = x_\alpha$ for $x \in \{a, b, c, d\}$. The action of $M_\mathbb{A}$ on $\mathbb{H}_n$ is given by $\sigma \cdot z = \alpha \cdot z$.

In [Shi93], Shimura defines a function $h_\sigma = h(\sigma, \cdot) : \mathbb{H}_n \to \mathbb{C}$ for any $\sigma \in \mathfrak{M}$ that acts as a factor of automorphy of weight $\frac{1}{2}$. This is done through the use of Weil's theory of theta series as mentioned above. In Lemma 1.1 of that paper, Shimura shows that one can identify M_∞ with the group of couples (α, g), where $\alpha \in G_\infty$ and $g : \mathbb{H}_n \to \mathbb{C}$ is holomorphic, that satisfy

$$g(z)^2 = \zeta |c_\alpha z + d_\alpha|,$$

for some constant $\zeta \in \mathbb{T}$. One has that C_p^θ is precisely the group $Ps(X_p, L_p)$ found in the 36th paragraph of the paper [Wei64] of Weil, yielding a lift

$$r_p : C_p^\theta \to M_p.$$

This previous point is the salient one as it allows Shimura, in Theorem 1.2 of [Shi93], to then construct h_σ by the transformation formulae of certain theta series for $\sigma \in \mathfrak{M}$. In this theorem, several properties of this function are proven, and we state the following relevant three:

$$h(\sigma, z)^2 = \zeta j(\mathrm{pr}(\sigma), z) \text{ for a constant } \zeta = \zeta(\sigma) \in \mathbb{T}, \tag{2.1.1}$$

$$h(r_P(\gamma), z) = |\det(d_\gamma)_\infty|_\mathbb{A}^{\frac{1}{2}} \text{ if } \gamma \in P_\mathbb{A}, \tag{2.1.2}$$

$$h(\rho\sigma\tau, z) = h(\rho, z) h(\sigma, \tau z) h(\tau, z) \text{ if } \mathrm{pr}(\rho) \in P_\mathbb{A} \text{ and } \mathrm{pr}(\tau) \in C^\theta. \tag{2.1.3}$$

Note that the third property is a "weak" automorphy property, which will not be sufficient for the Hecke theory we detail in the next section. But what actually is this function h? The theta series

$$\theta(z) := \sum_{x \in \mathbb{Q}^n} e(\tfrac{1}{2}x^T zx)$$

satisfies, by Proposition 1.3 of [Shi93],

$$\theta(\alpha \cdot z) = h(\alpha, z)\theta(z),$$

for any $\alpha \in G \cap \mathfrak{M}$. Therefore, in the $n = 1$ case, we have

$$h(\alpha, z) = \varepsilon_{d_\alpha}^{-1} \left(\frac{2c_\alpha}{d_\alpha}\right) (c_\alpha z + d_\alpha)^{\frac{1}{2}},$$

found in (4.40) of [Shi12], where

$$\varepsilon_{d_\alpha} = \begin{cases} 1 & \text{if } d_\alpha \equiv 1 \pmod 4) \\ i & \text{if } d_\alpha \equiv 3 \pmod 4). \end{cases}$$

For general n we restate the following proposition found in [Shi93, p. 1026]:

Proposition 2.1.2. *Suppose that $\alpha \in G \cap P_{\mathbb{A}} D[2, 2]$ and, according to the decomposition $G_{\mathbb{A}} = P_{\mathbb{A}} D[1, 1]$, write $\alpha = \rho w$ with $\rho \in P_{\mathbb{A}}$ and $w \in D[1, 1]$; we have*

$$h(\alpha, z)^2 = \operatorname{sgn}(|d_\alpha|) \left(\frac{-4}{|d_w|\mathbb{Z}}\right) (c_\alpha z + d_\alpha),$$

where $\left(\frac{-4}{\mathfrak{a}}\right)$ is the quadratic ideal character corresponding to $\mathbb{Q}(\sqrt{-1})/\mathbb{Q}$.

If k is a half-integral weight then put $[k] := k - \frac{1}{2} \in \mathbb{Z}$; if ℓ is an integral weight then put $[\ell] := \ell$. The factors of automorphy of half-integral weights k and integral weights ℓ are subsequently given as

$$\begin{aligned} j^k_\sigma(z) &: = h_\sigma(z) j(\mathrm{pr}(\sigma), z)^{[k]}, \\ j^\ell_\alpha(z) &: = j(\alpha, z)^\ell, \end{aligned}$$

where $\sigma \in \mathfrak{M}$, $\alpha \in G_{\mathbb{A}}$, and $z \in \mathbb{H}_n$. Given $\kappa \in \frac{1}{2}\mathbb{Z}$, a function $f : \mathbb{H}_n \to \mathbb{C}$, and an element $\xi \in G_{\mathbb{A}}$ or $\mathfrak{M}$ according as $\kappa \in \mathbb{Z}$ or not, we define a new function

$$(f||_\kappa \xi)(z) := j^\kappa_\xi(z)^{-1} f(\xi \cdot z).$$

The operator $||_\kappa$ is called the *slash operator.*

Definition 2.1.3. Let κ be an integral or half-integral weight and let $\Gamma \leq G$ be a congruence subgroup so that $\Gamma \leq \mathfrak{M}$ if $\kappa \notin \mathbb{Z}$. We denote by $C^\infty_\kappa(\Gamma)$ the complex vector space of C^∞ functions $f : \mathbb{H}_n \to \mathbb{C}$ such that $f||_\kappa \alpha = f$ for any $\alpha \in \Gamma$. Let $\mathcal{M}_\kappa(\Gamma) \subseteq C^\infty_\kappa(\Gamma)$ denote the subspace of holomorphic functions (with the additional holomorphy at cusps condition if $n = 1$). The elements of $\mathcal{M}_\kappa(\Gamma)$ are called *modular forms of weight κ and level Γ*; if $\kappa \notin \mathbb{Z}$ they are called *metaplectic modular forms.*

Modular forms in $\mathcal{M}_\kappa(\Gamma)$ have Fourier expansions summing over symmetric matrices $\tau \in M_n(\mathbb{Q})$ such that $\tau \geq 0$. If, for all $\xi \in G_{\mathbb{A}}$ or $\mathfrak{M}$ according as $\kappa \in \mathbb{Z}$ or not, the Fourier expansion of $f||_\kappa \xi$ sums over $\tau > 0$ then we call f a *cusp form.* The subspace of cusp forms of weight κ and level Γ is denoted $\mathcal{S}_\kappa(\Gamma)$.

Write

$$\mathcal{M}_\kappa := \bigcup_\Gamma \mathcal{M}_\kappa(\Gamma), \qquad \mathcal{S}_\kappa = \bigcup_\Gamma \mathcal{S}_\kappa(\Gamma),$$

where the unions are taken over all congruence subgroups in G (that are contained in $\mathfrak{M}$ if $\kappa \notin \mathbb{Z}$).

Definition 2.1.4. A *Hecke character* φ of $\mathbb{Q}$ is a continuous homomorphism

$$\varphi : \mathbb{I}_{\mathbb{Q}}/\mathbb{Q}^\times \to \mathbb{T},$$

whose restrictions to $\mathbb{Q}_p^\times$, $\mathbb{Q}_\infty^\times$, and $\mathbb{Q}_{\mathbf{f}}^\times$ are denoted $\varphi_p, \varphi_\infty$, and $\varphi_{\mathbf{f}}$ respectively. Each such Hecke character corresponds to a primitive *ideal* character φ^* given by

$$\varphi^* = \prod_{p \nmid \mathfrak{f}} \varphi_p,$$

where the integral ideal $\mathfrak{f}$ is the conductor of φ, and put $\varphi_{\mathfrak{a}} := \prod_{p|\mathfrak{a}} \varphi_p$ for any integral ideal $\mathfrak{a}$. There exist $t \in \mathbb{Z}$ and $\nu \in \mathbb{R}$ such that $\varphi_\infty(x) = \operatorname{sgn}(x_\infty)^t |x_\infty|^{i\nu}$, and we say that φ is *normalised* if $\nu = 0$. All Hecke characters that we consider are normalised. Let $\mathbb{Q}(\varphi)$ be the field extension formed by adding all values of φ. For an integer $r \geq 1$ define the r-degree Gauss sum of φ by

$$G_r(\varphi) := \sum_{a \in M_r(\mathbb{Z}/N(\mathfrak{f})\mathbb{Z})} \varphi_{\mathfrak{f}}^{-1}(|a|) e\left(\frac{\operatorname{tr}(a)}{N(\mathfrak{f})}\right), \tag{2.1.4}$$

and we put $G(\varphi) = G_1(\varphi)$.

Take $\kappa \in \frac{1}{2}\mathbb{Z}$, a fractional ideal $\mathfrak{b}$ and an integral ideal $\mathfrak{c}$. For the rest of this section, we make the key assumption that if $\kappa \notin \mathbb{Z}$ then

$$\begin{aligned} \mathfrak{b}^{-1} &\subseteq 2\mathbb{Z} \\ \mathfrak{b}\mathfrak{c} &\subseteq 2\mathbb{Z}; \end{aligned}$$

note that these conditions imply $\mathfrak{c} \subseteq 4\mathbb{Z}$. If $\kappa \in \mathbb{Z}$ then take *any* fractional ideal $\mathfrak{b}$ and integral ideal $\mathfrak{c}$. Put $\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$, the above assumption ensures that $\Gamma \leq \mathfrak{M}$ in the half-integral weight case. Now take a normalised Hecke character ψ of $\mathbb{Q}$ with the following two properties:

$$\psi_p(a) = 1 \text{ if } a \in \mathbb{Z}_p^\times \text{ and } a \in 1 + \mathfrak{c}_p, \tag{2.1.5}$$

$$\psi_\infty(x)^n = \operatorname{sgn}(x_\infty)^{n[\kappa]}. \tag{2.1.6}$$

The property of (2.1.5) guarantees that ψ_p is unramified for every $p \nmid \mathfrak{c}$. Modular forms of weight $\kappa \in \frac{1}{2}\mathbb{Z}$, level $\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$, and Hecke character ψ satisfying properties (2.1.5), (2.1.6) above are the fundamental objects of study in this thesis, the spaces of which are defined as

$$\begin{aligned} C^\infty_\kappa(\Gamma,\psi) &:= \{f : \mathbb{H}_n \to \mathbb{C} \in C^\infty \mid f||_\kappa\gamma = \psi_{\mathfrak{c}}(|a_\gamma|)f, \text{ for all } \gamma\in\Gamma\},\\ \mathcal{M}_\kappa(\Gamma,\psi) &:= \mathcal{M}_\kappa \cap C^\infty_\kappa(\Gamma,\psi),\\ \mathcal{S}_\kappa(\Gamma,\psi) &:= \mathcal{S}_\kappa \cap \mathcal{M}_\kappa(\Gamma,\psi). \end{aligned}$$

For any number field K, let $\mathcal{X}_\kappa(\Gamma,\psi,K)$ denote the space of forms in $\mathcal{X}_\kappa(\Gamma,\psi)$ whose Fourier coefficients all lie in K, where $\mathcal{X}\in\{\mathcal{M},\mathcal{S}\}$. Two key examples of modular forms, which play key roles throughout this thesis, are the non-holomorphic Siegel Eisenstein series and the theta series.

Eisenstein series. If ψ is a Hecke character that satisfies (2.1.5) and

$$\psi_\infty(x) = \operatorname{sgn}(x_\infty)^{[\kappa]}, \tag{2.1.7}$$

which condition differs from that of (2.1.6) in that the parity of the character is controlled by $[\kappa]$ if n is even. Recall the notation $\Delta(z) = |\mathfrak{Im}(z)|$ and $\mu(\gamma,z) = c_\gamma z + d_\gamma$ for any $\gamma\in G$ and $z\in\mathbb{H}_n$. The Eisenstein series is defined on variables $z\in\mathbb{H}_n$ and $s\in\mathbb{C}$ by the sum

$$E_\kappa(z,s;\psi,\Gamma) := \Delta(z)^{s-\frac{\kappa}{2}} \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \psi_{\mathfrak{c}}(|a_\gamma|) j^\kappa_\gamma(z)^{-1}|\mu(\gamma,z)|^{\kappa-2s}, \tag{2.1.8}$$

which is convergent on the half-plane $\mathfrak{Re}(s) > \frac{n+1}{2}$ [Shi00, p. 133], has meromorphic continuation to all of $s\in\mathbb{C}$ by a functional equation in $s\mapsto\frac{n+1}{2}-s$, and belongs to $C^\infty_\kappa(\Gamma,\psi^{-1})$. For $n>1$ other kinds of non-holomorphic Eisenstein series also exist, such as the Klingen Eisenstein series. These are defined and used in Section 2.3.2.

Theta series. Let $\tau\in M_n(\mathbb{Q})$ be symmetric and let $\mathfrak{t}$ be an integral ideal of $\mathbb{Z}$ satisfying $h^T(2\tau)^{-1}h\in 4\mathfrak{t}^{-1}$ for any $h\in\mathbb{Z}^n$. As an example, if $2\tau\in M_n(\mathbb{Z})$, then $\mathfrak{t} = 4|2\tau|\mathbb{Z}$ works. Take $\mu\in\{0,1\}$ and a Hecke character χ of conductor $\mathfrak{f}$ such that $\chi_\infty(x)^n = \operatorname{sgn}(x_\infty)^{n\mu}$. The theta series is defined by

$$\theta^{(\mu)}_\chi(z;\tau) = \theta_\chi(z) := \sum_{x\in M_n(\mathbb{Z})} (\chi_\infty\chi^*)^{-1}(|x|)|x|^\mu e(\operatorname{tr}(x^T\tau xz)), \tag{2.1.9}$$

where we understand that $(\chi_\infty\chi^*)(0) = 1$ if $\mathfrak{f}=\mathbb{Z}$ (otherwise it is zero) and recall that $\operatorname{tr}(q)$ denotes the trace of a matrix q. By Proposition 6.2 of [Shi96] this belongs to $\mathcal{M}_{\frac{n}{2}+\mu}(\Gamma[2,2\mathfrak{t}\mathfrak{f}^2],\chi^{-1}\rho_\tau)$, where ρ_τ is the quadratic character corresponding to the extension $\mathbb{Q}(i^{[n/2]}\sqrt{|2\tau|})/\mathbb{Q}$, and it has coefficients in $\mathbb{Q}(\chi)$.

Understand $\operatorname{pr} = \operatorname{id}$ if $\kappa\in\mathbb{Z}$. Setting $D := D[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}]$, we have $G_{\mathbb{A}} = GD$ by strong approximation (see [Bum96, Theorem 3.3.1]) and this holds for all $\mathfrak{b}$ and $\mathfrak{c}$. If $f\in\mathcal{M}_\kappa(\Gamma,\psi)$

we can define its adelisation $f_{\mathbb{A}} : \mathrm{pr}^{-1}(G_{\mathbb{A}}) \to \mathbb{C}$ by

$$f_{\mathbb{A}}(x) = \psi_{\mathfrak{c}}(|d_w|)(f||_{\kappa}w)(\mathbf{i}),$$

where $x = \alpha w$ for $\alpha \in G$, $w \in \mathrm{pr}^{-1}(D)$, and $\mathbf{i} = iI_n \in \mathbb{H}_n$. We have the following property:

$$f_{\mathbb{A}}(\alpha x w) = \psi_{\mathfrak{c}}(|d_w|)j_w^{\kappa}(\mathbf{i})^{-1}f_{\mathbb{A}}(x), \tag{2.1.10}$$

if $w \cdot \mathbf{i} = \mathbf{i}$, $w \in \mathrm{pr}^{-1}(D)$, and $\alpha \in G$ ([Shi95b, (1.16)]). This goes conversely so that the space $\mathcal{M}_{\kappa}(\Gamma, \psi)$ is in correspondence with the space of all functions $\mathrm{pr}^{-1}(G_{\mathbb{A}}) \to \mathbb{C}$ satisfying (2.1.10) via this adelisation map, [Shi95b, p. 26].

To give the precise Fourier expansions of these forms, define the following sets of symmetric matrices:

$$\begin{aligned} S &:= \{\xi \in M_n(\mathbb{Q}) \mid \xi^T = \xi\}, & S_+ &:= \{\xi \in S \mid \xi \geq 0\}, \\ S^{\triangledown} &:= \{\xi \in S \mid \xi_{ii} \in \mathbb{Z}, \xi_{ij} \in \tfrac{1}{2}\mathbb{Z}, i < j\}, & S_+^{\triangledown} &:= S^{\triangledown} \cap S_+, \\ S(\mathfrak{r}) &:= S \cap M_n(\mathfrak{r}), & S_{\mathbf{f}}(\mathfrak{r}) &:= \prod_{p \in \mathbf{f}} S(\mathfrak{r}_p), \end{aligned}$$

for any fractional ideal $\mathfrak{r}$ of $\mathbb{Q}$. An element of $S^{\triangledown}$ is known as a *symmetric half-integral matrix.*

The Fourier expansion of $f_{\mathbb{A}}$ encodes not only the Fourier coefficients of f, but also all the Fourier coefficients of its translates $f||_{\kappa}\xi$, where $\xi \in G$ if $\kappa \in \mathbb{Z}$ and $\xi \in G \cap \mathfrak{M}$ if $\kappa \notin \mathbb{Z}$, thus unifying the expansions of f at all cusps into one expression. We recap this expansion below and give the traditional Fourier expansions of modular forms afterwards.

Theorem 2.1.5 (Shimura, [Shi95b], p. 27). *Let $\kappa \in \frac{1}{2}\mathbb{Z}$ and put $D = D[\mathfrak{b}^{-1}, \mathfrak{bc}]$ and $\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}]$ (assuming that both are contained in $\mathfrak{M}$ if $\kappa \notin \mathbb{Z}$). Let $f \in \mathcal{M}_{\kappa}(\Gamma, \psi)$, $q \in GL_n(\mathbb{A}_{\mathbb{Q}})$, and $s \in S_{\mathbb{A}}$. Then the Fourier expansion of $f_{\mathbb{A}}$ is given as*

$$f_{\mathbb{A}}\left(r_P\begin{pmatrix} q & s\tilde{q} \\ 0 & \tilde{q}\end{pmatrix}\right) = |q_{\infty}|^{[\kappa]}\|q_{\infty}\|^{\kappa-[\kappa]}\sum_{\tau \in S_+} c(\tau, q; f)e_{\infty}(\mathrm{tr}(\mathbf{i}q^T\tau q))e_{\mathbb{A}}(\mathrm{tr}(\tau s)),$$

where $c(\tau, q; f) = c_f(\tau, q) \in \mathbb{C}$ and recall $\tilde{q} = (q^T)^{-1}$. Furthermore, the coefficients c_f obey:

(i) $c_f(\tau, q) \neq 0$ *only if* $e_{\mathbb{A}}(\mathrm{tr}(q^T\tau q s)) = 1$ *for all* $s \in S_{\mathbf{f}}(\mathfrak{b}^{-1})$*;*

(ii) $c_f(\tau, q) = c_f(\tau, q_{\mathbf{f}})$*;*

(iii) $c_f(b^T\tau b, q) = |b|^{[\kappa]}\|b\|^{\kappa-[\kappa]}c_f(\tau, bq)$ *for any* $b \in GL_n(\mathbb{Q})$*;*

(iv) $\psi_{\mathbf{f}}(|a|)c_f(\tau, qa) = c_f(\tau, q)$ *for any* $\mathrm{diag}[a, \tilde{a}] \in D$*;*

(v) if $\beta \in G \cap \mathrm{diag}[r, \tilde{r}]D$*,* $r \in GL_n(\mathbb{A}_{\mathbb{Q}})$*, and* $z \in \mathbb{H}_n$*, then*

$$j_{\beta}^{\kappa}(\beta^{-1}z)f(\beta^{-1}z) = \psi_{\mathfrak{c}}(|d_{\beta}r|)\sum_{\tau \in S_+} c_f(\tau, r)e(\mathrm{tr}(\tau z)).$$

The coefficients $c_f(\tau,1)$ are the usual Fourier coefficients of f in the following sense: by (i) of the above theorem, the modular form $f \in \mathcal{M}_\kappa(\Gamma,\psi)$ has Fourier expansion

$$f(z) = \sum_{\tau \in S_+} c_f(\tau,1) e(\operatorname{tr}(\tau z)),$$

where $c_f(\tau,1) \neq 0$ only if $\tau \in N(\mathfrak{b}) S_+^\triangledown$. By Theorem 2.1.5 (v) above, the coefficients $c_f(\tau,r)$ are the Fourier coefficients of f at the cusp corresponding to r.

If $F \in C_\kappa^\infty(\Gamma,\psi)$ then it has a Fourier expansion of the form

$$F(z) = \sum_{\tau \in S} c_F(\tau,y) e(\operatorname{tr}(\tau x)),$$

where $z = x+iy$ and the coefficients $c_F(\tau,y)$ are smooth functions of y with values in $\mathbb{C}$. As above, $c_F(\tau,y)$ is identically zero unless $\tau \in N(\mathfrak{b}) S^\triangledown$.

Definition 2.1.6. The space $\operatorname{Aut}(\mathbb{C})$ acts on $\mathcal{M}_\kappa(\Gamma)$ as follows. If $f \in \mathcal{M}_\kappa(\Gamma)$ and $\sigma \in \operatorname{Aut}(\mathbb{C})$, then $f^\sigma \in \mathcal{M}_\kappa(\Gamma)$ is the form given by

$$f^\sigma(z) := \sum_{\tau \in S_+} c_f(\tau,1)^\sigma e(\operatorname{tr}(\tau z)).$$

Remark 2.1.7. That this is an action at all is non-trivial. See p. 35 and (Q1) in [Shi00] for a discussion on this, and Chapters 9 and 10 for the proof.

We end this section with some final fundamental definitions. Define: differentials on $n \times n$ symmetric matrices by

$$dx = \bigwedge_{p \le q} dx_{pq}, \qquad dy = \bigwedge_{p \le q} dy_{pq}, \qquad d^\times y = |y|^{-\frac{n+1}{2}} dy,$$
$$dz = dxdy, \qquad d^\times z = |y|^{-n-1} dxdy = |y|^{-\frac{n+1}{2}} dx d^\times y;$$

the sets of matrices

$$X := \{x \in M_n(\mathbb{R}) \mid x^T = x, -\tfrac{1}{2} \le x_{ij} \le \tfrac{1}{2} \text{ for all } i,j\}, \tag{2.1.11}$$
$$Y := \{y \in M_n(\mathbb{R}) \mid y^T = y, y > 0\}; \tag{2.1.12}$$

and the generalised Gamma function

$$\Gamma_n(s) := \int_Y |y|^s e^{-\operatorname{tr}(y)} d^\times y = \pi^{\frac{n(n-1)}{4}} \prod_{i=0}^{n-1} \Gamma\left(s - \tfrac{i}{2}\right). \tag{2.1.13}$$

Consider $\mathfrak{b}$ fixed in the definition of $\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}]$, so that Γ depends only on $\mathfrak{c}$, and let ψ be a normalised Hecke character satisfying (2.1.5) and (2.1.6). Then for any two $f,g \in C_\kappa^\infty(\Gamma,\psi)$ we define the Petersson inner product by

$$\langle f,g \rangle_{\mathfrak{c}} = \langle f,g \rangle := \operatorname{Vol}(\Gamma \backslash \mathbb{H}_n)^{-1} \int_{\Gamma \backslash \mathbb{H}_n} f(z) \overline{g(z)} \Delta(z)^\kappa d^\times z. \tag{2.1.14}$$

This integral is convergent whenever one of f,g belongs to $\mathcal{S}_\kappa(\Gamma,\psi)$.

2.2 Hecke theory

Notation. k – half-integral weight; $[k] = k - \frac{1}{2} \in \mathbb{Z}$.
$\mathfrak{b}$ – fractional ideal of $\mathbb{Q}$; $\mathfrak{c}$ – integral ideal of $\mathbb{Z}$.
$(\mathfrak{b}^{-1}, \mathfrak{bc}) \subseteq 2\mathbb{Z} \times 2\mathbb{Z}$.
$D = D[\mathfrak{b}^{-1}, \mathfrak{bc}]$; $\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}]$.
$P = \{\alpha \in Sp_n(\mathbb{Q}) \mid c_\alpha = 0\}$; $r_P : P_{\mathbb{A}} \to M_{\mathbb{A}}$ – lift.
$S = \{\xi \in M_n(\mathbb{Q}) \mid \xi^T = \xi\}$.
$\tilde{q} = (q^T)^{-1}$ for any invertible matrix q.

We have already seen some major differences in this setting to that of integral-weight forms – namely the factor h_σ having only a "weak" automorphy property and congruence subgroup restriction. Both of these differences have ramifications in constructing a theory of Hecke operators for metaplectic forms, with the former requiring an extension of h_σ to have a "strong" automorphic property and the latter causing natural restrictions to the operators we consider.

Definition 2.2.1. Two subgroups H_1 and H_2 of a multiplicative group H are *commensurable* if $H_1 \cap H_2$ has finite index in both H_1 and H_2; commensurability is transitive.

Let Δ be a subgroup of H; its commensurator is the set of all $h \in H$ such that $h^{-1}\Delta h$ and Δ are commensurable. Let Ξ be a subsemigroup of H. Then (Δ, Ξ) is called a *Hecke pair* if Ξ is contained in the commensurator of Δ and if $\Delta\Xi = \Xi\Delta = \Xi$.

If (Δ, Ξ) is a Hecke pair, let $\mathcal{R}(\Delta, \Xi)$ denote the ring of formal finite sums $\sum_\xi c_\xi \Delta\xi\Delta$, with $c_\xi \in \mathbb{C}$ and $\xi \in \Xi$. Each double coset in this sum has a finite decomposition into single right cosets, see [And79, p. 78]. For the law of multiplication in this ring see [And79, pp. 78 – 79].

In order to define the appropriate Hecke ring acting on modular forms of half-integral weight, we first define some groups:

$$\mathsf{O}_p := GL_n(\mathbb{Z}_p), \qquad \mathsf{O} := \prod_p GL_n(\mathbb{Z}_p),$$

$$\mathsf{X}_p := M_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p), \qquad \mathsf{X} := GL_n(\mathbb{Q})_{\mathbf{f}} \cap \prod_p \mathsf{X}_p,$$

$$Z_0 := \{\mathrm{diag}[\tilde{q}, q] \mid q \in \mathsf{X}\}, \qquad Z := D[2,2] Z_0 D[2,2],$$

and certain metaplectic lifts

$$\mathfrak{D}[2,2] := \mathrm{pr}^{-1}(D[2,2]), \qquad \mathfrak{D} := \{\alpha \in \mathfrak{D}[2,2] \mid \mathrm{pr}(\alpha) \in G_{\mathbf{f}} \cap D\},$$

$$\mathfrak{Z} := \mathrm{pr}^{-1}(Z), \qquad \mathfrak{Z}_0 := \{\alpha \in \mathfrak{Z} \mid \mathrm{pr}(\alpha) \in G_{\mathbf{f}} \cap DZ_0D\},$$

$$\widehat{\mathfrak{Z}}_0 := \{(\alpha, t) \mid t \in \mathbb{T}, \alpha \in \mathfrak{Z}_0\}, \qquad \widehat{\mathfrak{D}} := \{(\alpha, 1) \in \widehat{\mathfrak{Z}}_0 \mid \alpha \in \mathfrak{D}\}.$$

For any prime p we use the subscript p to denote the pth local component of any of the above adelic groups.

To define the action of Hecke operators on modular forms later on, it is necessary to extend the half-integral weight factor of automorphy h to all of $\mathfrak{Z}$. This extension should have stronger automorphic properties than the aforementioned factor h, which had the weak property (2.1.3). In [Shi95b, p. 32] Shimura defines a new factor of automorphy J^k, which extends the original j^k to $\mathfrak{Z}$ and which has the desired strong automorphic properties.

Definition 2.2.2. Let $\alpha = \xi_1\mathfrak{z}\xi_2 \in \mathfrak{Z}$, where $\xi_i \in \mathfrak{D}[2,2]$ and $\mathrm{pr}(\mathfrak{z}) \in Z_0$. For $z \in \mathbb{H}_n$ define

$$J^k(\alpha, z) := j^k(\xi_1\xi_2, z).$$

The factor J^k is well defined, [Shi95b, p. 32], and satisfies the following three properties

$$J^k(\xi, z) = j^k(\xi, z) \qquad \text{if } \xi \in \mathfrak{D}[2,2], \tag{2.2.1}$$

$$J^k(\xi\alpha\eta, z) = J^k(\xi, \alpha\eta z)J^k(\alpha, \eta z)J^k(\eta, z) \qquad \text{if } \alpha \in \mathfrak{Z} \text{ and } \xi, \eta \in \mathfrak{D}[2,2], \tag{2.2.2}$$

$$J^k(\alpha, z) = j^{[k]}(\mathrm{pr}(\alpha), z)J^{\frac{1}{2}}(\alpha, z) \qquad \text{for any } \alpha \in \mathfrak{Z}. \tag{2.2.3}$$

These properties are obtained immediately from the definition; pay special emphasis on (2.2.2) which gives a satisfactory factor of automorphy for the theory of Hecke operators.

Define a law of multiplication in $\widehat{\mathfrak{Z}}_0$ by

$$(\alpha, t)(\alpha, t') := \left(\alpha\alpha', tt'\tfrac{J(\alpha)J(\alpha')}{J(\alpha\alpha')}\right),$$

where $J(\alpha) := J^{\frac{1}{2}}(\alpha, \mathbf{i})$ for $\alpha \in \mathfrak{Z}$. Since $\mathrm{pr}(\alpha) \in G_{\mathbf{f}} \cap DZ_0D$ we can write $\alpha = \xi_1\mathfrak{z}\xi_2$, where $\mathrm{pr}(\xi_i) \in G_{\mathbf{f}} \cap D[2,2]$ are each trivial at the infinite place. Therefore $h(\xi_1\xi_2, \mathbf{i}) = 1$, $J(\alpha) \in \mathbb{T}$, and the above law of multiplication makes sense; essentially the notation $J(\alpha)$ is referring to an extension of the element $\zeta \in \mathbb{T}$ given in (2.1.1). Notice by the property of (2.2.2) above that $J(\alpha\alpha') = J(\alpha)J(\alpha')$ if either $\alpha \in \mathfrak{D}$ or $\alpha' \in \mathfrak{D}$.

The Hecke ring we consider is $\mathcal{R}(\widehat{\mathfrak{D}}, \widehat{\mathfrak{Z}}_0)$, and take $f \in \mathcal{M}_k(\Gamma, \psi)$. The action of the element $T := \widehat{\mathfrak{D}}(\alpha, t)\widehat{\mathfrak{D}}$ of this Hecke ring on the adelisation $f_{\mathbb{A}} : M_{\mathbb{A}} \to \mathbb{C}$ is given by first decomposing into finitely many single cosets $T = \bigsqcup_\beta \widehat{\mathfrak{D}}(\beta, t)$, where $\beta \in \mathfrak{Z}_0$, and subsequently putting

$$(f_{\mathbb{A}}|T_\psi)(x) := \sum_\beta \psi_{\mathfrak{c}}(|a_\beta|)^{-1}\bar{t}J(\beta)^{-1}f_{\mathbb{A}}(x\beta^{-1}). \tag{2.2.4}$$

If $q \in \mathsf{X}$ then write $T_{q,\psi} = T_\psi$ whenever $\alpha = r_P(\mathrm{diag}[\tilde{q}, q])$ and $t = 1$. In this case we can give the explicit action on f itself by considering finite decompositions of global double cosets of the form

$$G \cap (D\,\mathrm{diag}[\tilde{q}, q]D) = \Gamma\alpha\Gamma = \bigsqcup_\beta \Gamma\beta,$$

where $\alpha, \beta \in G \cap Z$, and by putting

$$(f|T_{q,\psi})(z) := \sum_\beta \psi_{\mathfrak{c}}(|a_\beta|)^{-1}J^k(\beta, z)^{-1}f(\beta\cdot z); \tag{2.2.5}$$

by [Shi95b, p. 41] we have $f_{\mathbb{A}}|T_{q,\psi} = (f|T_{q,\psi})_{\mathbb{A}}$ and these two notions indeed match up.

Lemma 2.6 of [Shi94] tells us what these single coset representatives look like; we restate it here.

Lemma 2.2.3. *For any $g, h \in \mathsf{X}$ let $g\mathbb{Z}^n + h\mathbb{Z}^n$ denote the $\mathbb{Z}$-lattice in $\mathbb{Q}$ such that $(g\mathbb{Z}^n + h\mathbb{Z}^n)_p = g_p\mathbb{Z}_p^n + h_p\mathbb{Z}_p^n$ for all primes p and define the sets*

$$W_{\mathfrak{c}} := \{(g,h) \in \mathsf{X}^2 \mid g\mathbb{Z}^n + h\mathbb{Z}^n = \mathbb{Z}^n, h_p \in \mathsf{O}_p \text{ if } p \mid \mathfrak{c}\},$$
$$S' := \{\sigma \in S_{\mathbf{f}} \mid \sigma_p \in S(\mathfrak{b}^{-1})_p \text{ if } p \mid \mathfrak{c}\}.$$

Let $\mathsf{O}\backslash W_{\mathfrak{c}}$ denote the right coset representatives under simultaneous multiplication – i.e. $u \cdot (g,h) := (ug, uh)$, where $u \in \mathsf{O}$ and $(g,h) \in W_{\mathfrak{c}}$. Let $W_{\mathfrak{c}}/(\mathsf{O} \times 1)$ denote the left coset representatives under multiplication on the first argument only – i.e. $(g,h)\cdot(u,1) := (ug,h)$. A complete set of representatives for $\mathfrak{D}\backslash\mathfrak{Z}_0$ is given by

$$\left\{ \begin{pmatrix} g^{-1}h & g^{-1}\sigma\tilde{h} \\ 0 & g^T\tilde{h} \end{pmatrix} \middle| (g,h) \in \mathsf{O}\backslash W_{\mathfrak{c}}/(\mathsf{O}\times 1), \sigma \in S'/gS_{\mathbf{f}}(\mathfrak{b}^{-1})g^T \right\}.$$

Definition 2.2.4. Form the factor ring

$$\mathcal{R}_0 := \mathcal{R}(\widehat{\mathfrak{D}}, \widehat{\mathfrak{Z}}_0)/\langle \widehat{\mathfrak{D}}(\alpha,1)\widehat{\mathfrak{D}} - t\widehat{\mathfrak{D}}(\alpha,t)\widehat{\mathfrak{D}} \mid (\alpha,t) \in \widehat{\mathfrak{Z}}_0 \rangle, \tag{2.2.6}$$

and let $A_q \in \mathcal{R}_0$ denote the image of $T_{q,\psi}$ under projection to $\mathcal{R}_0$. The factor ideal acts trivially on $\mathcal{M}_k(\Gamma,\psi)$ by (2.2.4), thus giving an action $f|A_q$ of $A_q \in \mathcal{R}_0$ on $f \in \mathcal{M}_k(\Gamma,\psi)$ by the same formulae of (2.2.4) and (2.2.5).

For any prime p denote by $\mathcal{R}_{0p}$ the subalgebra generated by A_q for all $q \in \mathsf{X}_p$.

An element $f \in \mathcal{S}_k(\Gamma,\psi)$ that is an eigenvector for all Hecke operators in $\mathcal{R}_0$ is called an *eigenform*.

Eigenforms and their L-functions carry a wealth of arithmetic information and they are a fundamental object of study for this thesis. The association of the standard L-function to an eigenform involves the determination of a bunch of non-zero complex numbers, known as *Satake parameters*, whose determination comes about through the Satake maps.

The Satake maps are local injective maps between $\mathcal{R}_{0p}$ and polynomials in $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$ and they are defined for all primes p via the composition of two maps

$$\omega_p := \omega_{0p} \circ \Phi_p : \mathcal{R}_{0p} \to \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}],$$

which will be defined below. It is well known that if $p \nmid \mathfrak{c}$ then this is an isomorphism between $\mathcal{R}_{0p}$ and $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]^{W_n}$, which is the ring of symmetric polynomials invariant under the Weyl transformations $x_i \mapsto x_i^{-1}; x_j \mapsto x_j$ for $j \neq i$ – see for example [And79, Lemma 1.2.2]. We adopt the following definitions from [Shi95b, pp. 41–42].

The map Φ_p. If $q \in \mathsf{X}_p$ then from Lemma 2.2.3 we have a decomposition of the form

$$D_p \operatorname{diag}[\tilde{q}, q] D_p = \bigsqcup_{x \in X} \bigsqcup_{s \in Y_x} \bigsqcup_{d \in R_x} D_p \alpha_{d,s}, \qquad \alpha_{d,s} = \begin{pmatrix} \tilde{d} & sd \\ 0 & d \end{pmatrix}, \tag{2.2.7}$$

with $X \subseteq GL_n(\mathbb{Q}_p), R_x \subseteq x\mathsf{O}_p$ representing the single coset decomposition $\mathsf{O}_p\backslash\mathsf{O}_p x\mathsf{O}_p$, and $Y_x \subseteq S_p$. Recall the notation $J(\alpha) = J^{\frac{1}{2}}(\alpha, \mathbf{i})$ for $\alpha \in \mathfrak{Z}$ and extend to all of $\mathcal{R}_{0p}$ by $\mathbb{C}$-linearity the map

$$\Phi_p(A_q) := \sum_{d,s} J(r_P(\alpha_{d,s}))^{-1}\mathsf{O}_p d \in \mathcal{R}(\mathsf{O}_p, GL_n(\mathbb{Q}_p)).$$

The map $\Phi_p : \mathcal{R}_{0p} \to \mathcal{R}(\mathsf{O}_p, GL_n(\mathbb{Q}_p))$ is injective for all p, see [Shi95b, Lemma 4.3].

The map ω_{0p}. Note that any coset $\mathsf{O}_p d$ with $d \in GL_n(\mathbb{Q}_p)$ contains an upper triangular matrix of the form

$$\begin{pmatrix} p^{a_{d_1}} & \star & \cdots & \star \\ 0 & p^{a_{d_2}} & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p^{a_{d_n}} \end{pmatrix}, \tag{2.2.8}$$

with $a_{d_i} \in \mathbb{Z}$, and then define

$$\omega_{0p}(\mathsf{O}_p d) := \prod_{i=1}^{n} (p^{-i}x_i)^{a_{d_i}},$$

which, via decompositions $\mathsf{O}_p x\mathsf{O}_p = \sum_d \mathsf{O}_p d$ and $\mathbb{C}$-linearity, we extend to obtain the map $\omega_{0p} : \mathcal{R}(\mathsf{O}_p, GL_n(\mathbb{Q}_p)) \to \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$. This map is an isomorphism between $\mathcal{R}(\mathsf{O}_p, GL_n(\mathbb{Q}_p))$ and $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]^{\Sigma_n}$, which is the ring of all symmetric polynomials.

The above local maps are used to associate to a Hecke eigenform $f \in \mathcal{S}_k(\Gamma, \psi)$ an element in $\mathbb{C}^n$ for all primes p. Define a formal Dirichlet series with coefficients in $\mathcal{R}_{0p}$ by

$$\mathcal{T}_p := \sum_{m=0}^{\infty} A(p^m)t^m,$$

where $A(p^m) \in \mathcal{R}_{0p}$ is the sum of all $A_q \in \mathcal{R}_{0p}$ over $q \in \mathsf{O}_p\backslash\mathsf{X}_p/\mathsf{O}_p$ with $|q| = p^m$. This series acts on any $f \in \mathcal{M}_k(\Gamma, \psi)$ by letting it act coefficient-wise; denote this action $f|\mathcal{T}_p$. If f is an eigenform in the sense of Definition 2.2.4, with $f|A(p^m) = \Lambda(p^m)f$, we get

$$f|\mathcal{T}_p = \sum_{m=0}^{\infty} \Lambda(p^m)t^m f. \tag{2.2.9}$$

Theorem 4.4 of [Shi95b, p. 42] tells us that

$$\omega_p(\mathcal{T}_p) = \begin{cases} \displaystyle\prod_{i=1}^{n} (1 - p^n x_i t)^{-1} & \text{if } p \mid \mathfrak{c}, \\ \displaystyle\prod_{i=1}^{n} \frac{1 - p^{2i-1}t^2}{(1 - p^n x_i t)(1 - p^n x_i^{-1} t)} & \text{if } p \nmid \mathfrak{c}, \end{cases} \tag{2.2.10}$$

and subsequently we get the existence of an n-tuple $(\lambda_{p,1}, \dots, \lambda_{p,n}) \in \mathbb{C}^n$ such that

$$\sum_{m=0}^{\infty} \Lambda(p^m)t^m = \begin{cases} \prod_{i=1}^{n} (1 - p^n \lambda_{p,i} t)^{-1} & \text{if } p \mid \mathfrak{c}, \\ \prod_{i=1}^{n} \dfrac{1 - p^{2i-1}t^2}{(1 - p^n \lambda_{p,i} t)(1 - p^n \lambda_{p,i}^{-1} t)} & \text{if } p \nmid \mathfrak{c}. \end{cases} \tag{2.2.11}$$

Definition 2.2.5. Let $f \in \mathcal{S}_k(\Gamma, \psi)$ be an eigenform, where $\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}] \leq \mathfrak{M}$, and put

$$\psi^{\mathfrak{c}} := \frac{\psi}{\psi_{\mathfrak{c}}}.$$

Let $\chi : \mathbb{I}_{\mathbb{Q}}/\mathbb{Q}^\times \to \mathbb{T}$ be a normalised Hecke character in the sense of Definition 2.1.4 and let $(\lambda_{p,1}, \dots, \lambda_{p,n}) \in \mathbb{C}^n$ denote the Satake p-parameters of f for each prime p. The *standard L-function* associated to f is the following function in the complex variable s:

$$\begin{aligned} L_p(t) &:= \begin{cases} \prod_{i=1}^{n} (1 - p^n \lambda_{p,i} t) & \text{if } p \mid \mathfrak{c}, \\ \prod_{i=1}^{n} (1 - p^n \lambda_{p,i} t)(1 - p^n \lambda_{p,i}^{-1} t) & \text{if } p \nmid \mathfrak{c}, \end{cases} \\ L_\psi(s, f, \chi) &:= \prod_{p} L_p\left((\psi^{\mathfrak{c}} \chi^*)(p) p^{-s}\right)^{-1}. \end{aligned}$$

The Euler product defining $L_\psi(s, f, \eta)$ is absolutely convergent, and therefore non-zero, for $\mathfrak{Re}(s) > \frac{3n}{2} + 1$, see [Shi96, Theorem A, p. 332]. In addition, it can be meromorphically continued to the whole s-plane with finitely many poles. The location of these poles can be determined using the Rankin-Selberg integral expression [Shi96, (4.1)]; under additional assumptions this is Theorem B2 of [Shi96] and we restate an appropriate version of it here.

Theorem 2.2.6 (Shimura, [Shi96])**.** *Suppose that $n > 1$. Let $f \in \mathcal{S}_k(\Gamma^\star, 1)$, where*

$$\Gamma^\star := \left\{ \gamma \in \Gamma[\mathfrak{b}^{-1}\mathfrak{c}, \mathfrak{bc}] \mid a_\gamma \equiv 1 \pmod{\mathfrak{c}} \right\}$$

is analogous to the principal congruence subgroup. Let χ be a Hecke character of conductor $\mathfrak{f}$ and take $\varepsilon \in \{0, 1\}$ such that $\chi_\infty(x) = \operatorname{sgn}(x_\infty)^{[k]+\varepsilon}$. Then the standard L-function

$$L_{\mathfrak{c}}(s, f, \chi) := \prod_{p \nmid \mathfrak{c}} L_p\left(\chi^*(p) p^{-s}\right)^{-1},$$

with Euler factors at $\mathfrak{c}$ removed, is meromorphic on the whole s-plane with finitely many poles, all of which are simple. The L-function is entire if $\chi^2 \neq 1$. If $\chi^2 = 1$ then $L_{\mathfrak{c}}(s, f, \chi)$ has the following possible poles.

(i) If $k > n + \varepsilon$ then $L_{\mathfrak{c}}(s, f, \chi)$ is entire for half-integral weights.

(ii) If $k \leq n + \varepsilon$ then $L_{\mathfrak{c}}(s, f, \chi)$ may have poles only in the set

$$\{j + \tfrac{1}{2} \mid j \in \mathbb{Z}, n + 1 \leq j \leq \min\{2n - k + \varepsilon, \tfrac{3n}{2}\} + \tfrac{1}{2}, j \geq 2n + [k] - \varepsilon\}.$$

Remark 2.2.7. Theorem B1 of [Shi96] is a similar kind of result to Theorem 2.2.6 above but for $L_{\mathfrak{c}}(s, f, \chi)$ normalised by some Gamma factors (defined in [Shi95b, (6.4a–b)]) so that the proof of the above theorem follows directly from Theorem B1.

Weaker versions of Theorem B1 appeared in [Shi95a, Theorem 6.1] for integral weights, which was proved using the Rankin-Selberg method, and in [Shi95b, Theorem 6.1] for half-integral weights, which was proved using the *doubling method.* We mention only that the doubling method is another tool that can be used to prove similar results to those obtained using the Rankin-Selberg method, see Sections 7–8 of [Shi95b] for details. In either case, the poles are obtained from those of the Eisenstein series and those of the normalising Gamma factors. The improvements of both these results to give Theorem B1 of [Shi96] follow by increasing the non-vanishing range of the L-function to $\mathfrak{Re}(s) > \frac{3n}{2}+1$, see [Shi96, Theorem A]. Therefore the above theorem actually follows using the doubling method, not the Rankin-Selberg method. One can prove a similar result using the Rankin-Selberg method by removing the additional assumptions, however we must take into account possible poles of a quotient of Dirichlet L-functions $\frac{\Lambda_{\mathfrak{c}}}{\Lambda_{\mathfrak{y}}}$, which we define later in (2.4.1). We do this in the vector-valued case in Theorem D2 of Chapter 5.

2.3 Algebraic decomposition

The results of this chapter can also be found in Sect. 6 of [Mer18b].

Notation. k – half-integral weight; $[k] = k - \frac{1}{2} \in \mathbb{Z}$.
$\mathfrak{b}$ – fractional ideal of $\mathbb{Q}$; $\mathfrak{c}$ – integral ideal of $\mathbb{Z}$.
$(\mathfrak{b}^{-1}, \mathfrak{bc}) \subseteq 2\mathbb{Z} \times 2\mathbb{Z}$.
$D = D[\mathfrak{b}^{-1}, \mathfrak{bc}]$; $\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}]$.
$P = \{\alpha \in Sp_n(\mathbb{Q}) \mid c_\alpha = 0\}$; $r_P : P_{\mathbb{A}} \to M_{\mathbb{A}}$ – lift.
Superscript n on groups defined in Section 2.1 and Section 2.2
(G^n, $\mathcal{M}_k^n$, D_p^n, $\mathcal{R}_{0p}^n$ etc.) *when emphasis is necessary.*
$\mathsf{X}_p = M_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p)$; $\mathsf{O}_p = GL_n(\mathbb{Z}_p)$;
$\mathsf{X} = GL_n(\mathbb{Q})_{\mathbf{f}} \cap \prod_p \mathsf{X}_p$; $\mathsf{O} = \prod_p \mathsf{O}$.
$Z_0 = \{\mathrm{diag}[\tilde{q}, q] \mid q \in \mathsf{X}\}$; $Z = D[2,2]Z_0D[2,2]$; $\mathfrak{Z} = \mathrm{pr}^{-1}(Z)$.
$J(\alpha) = J^{\frac{1}{2}}(\alpha, \mathbf{i})$ for any $\alpha \in \mathfrak{Z}$.
$\tilde{q} = (q^T)^{-1}$ for any invertible matrix q.

That the space of modular forms decomposes into cusp forms and Eisenstein series is well known; it is not so obvious that this should preserve algebraicity of the Fourier coefficients of the forms involved and, should it indeed preserve this, it is even less clear to what degree this preservation of algebraicity occurs. The results of this section clarify this issue for the half-integral weight case.

To be more precise, we prove

$$\mathcal{M}_k(\mathscr{L}) = \mathcal{S}_k(\mathscr{L}) \oplus \mathcal{E}_k(\mathscr{L}), \tag{2.3.1}$$

where $\mathscr{L}/\mathbb{Q}$ is an algebraic field extension, $\mathcal{E}_k$ is the space of Eisenstein series, and $\mathcal{X}_k(\mathscr{L})$ for $\mathcal{X} \in \{\mathcal{M}, \mathcal{S}, \mathcal{E}\}$ denotes the modular forms in $\mathcal{X}_k$ whose Fourier coefficients lie in $\mathscr{L}$. The decomposition of (2.3.1) is known in the present setting when $\mathscr{L} = \overline{\mathbb{Q}}$, this was shown by Shimura in Theorem 27.16 of [Shi00]. We improve on this by detailing a precise field extension of $\mathbb{Q}$ for which (2.3.1) holds; the benefits of this precision will become apparent in Chapter 3 and will allow the algebraic determination of the full range of special values of the L-function.

To achieve this result we prove an analogue of Garrett's conjecture, of [Gar84], that if f has algebraic coefficients then its Klingen Eisenstein series $E(f)$ does too. That such a result should lead to the decomposition (2.3.1) is natural since $\mathcal{M}_k = \mathcal{S}_k \oplus \mathcal{E}_k$ is given precisely by writing f as the sum of Klingen Eisenstein series associated to it. The proof of Garrett's conjecture is achieved by a non-trivial extension of the methods of Harris in [Har81], whose setting is integral-weight and full-level Siegel modular forms. Later, in [Har84], Harris extends this result to more general automorphic forms that are associated to Shimura varieties which does not cover the present case.

- For the rest of this section, we use the superscript n on symbols already introduced which naturally depend on n, for example $G^n, \mathcal{M}_k^n, D_p^n, \mathcal{R}_{0p}^n$ and so on.

2.3.1 The Siegel Phi operator

The purpose of this preliminary subsection is to give a relation between the standard L-function of an n-degree form f and that of the $n-1$-degree form Φf, where Φ is the Siegel Phi operator. This type of relation has been studied before; for forms of integral weight and trivial character this was done by Zharkovskaya in [Zha74] and later for non-trivial character by Andrianov in [And79]. It has also been established by Hayashida [Hay03] for half-integral weight Siegel modular forms, using the Hecke ring construction of Zhuravlev in [Zhu84], [Zhu85] and results of Oh-Koo-Kim [OOK89], but it is not so clear how their setting translates to that of the present.

For a real variable ρ the Siegel Phi operator is defined as

$$\begin{aligned} \Phi : \mathcal{M}_k^n &\to \mathcal{M}_k^{n-1} \\ f(z) &\mapsto \lim_{\rho \to \infty} f \begin{pmatrix} w & 0 \\ 0 & i\rho \end{pmatrix}, \end{aligned}$$

for variables $z \in \mathbb{H}_n, w \in \mathbb{H}_{n-1}$.

Take a prime p such that $p \nmid \mathfrak{c}$. Define a map $\Psi(\cdot, u) : \mathcal{R}_{0p}^n \to \mathcal{R}_{0p}^{n-1}[u^{\pm}]$, for an independent variable u, as follows. Consider the generators $A_q \in \mathcal{R}_{0p}^n$, for $q \in \mathsf{X}_p$, with $D_p^n \operatorname{diag}[\tilde{q}, q] D_p^n$ having the decomposition of (2.2.7) and each d in the decomposition having the form in (2.2.8), recall this means that the ith diagonal entry of d is $p^{a_{d_i}}$ for $a_{d_i} \in \mathbb{Z}$. Then put

$$\Psi(A_q, u) := \sum_{x,d,s} J(r_P(\alpha_{d',s'}))J(r_P(\alpha_{d,s}))^{-1}(up^{-n})^{a_{d_n}} D_p^{n-1} \begin{pmatrix} \tilde{d}' & s'd' \\ 0 & d' \end{pmatrix},$$

where a' denotes the upper-left $n-1$ block of any $n\times n$ matrix a and recall $J(\alpha) = J^{\frac{1}{2}}(\alpha, \mathbf{i})$ for any $\alpha \in \mathfrak{Z}$. Extend this to all of $\mathcal{R}_{0p}^n$ by $\mathbb{C}$-linearity. The map $(\omega_p^{n-1}\times 1)$ on $\mathcal{R}_{0p}^{n-1}[u^\pm]$ acts as ω_p^{n-1} on $\mathcal{R}_{0p}^{n-1}$ and as the identity on u. Then

$$(\omega_p^{n-1}\times 1)(\Psi(A_q,u)) = \sum_{d,s} J(r_P(\alpha_{d,s}))^{-1}(up^{-n})^{a_{d_n}}\prod_{i=1}^{n-1}(p^{-i}x_i)^{a_{d_i}}.$$

So by defining $\phi_{n,u}(x_i) = x_i$ for $1\le i\le n-1$ and $\phi_{n,u}(x_n) = u$ and extending $\mathbb{C}$-linearly to all of $\mathbb{C}[x_1^\pm,\dots,x_n^\pm]$, we get the commuting square

$$\begin{array}{ccc} \mathcal{R}_{0p}^n & \xrightarrow{\omega_p^n} & \mathbb{C}[x_1^\pm,\dots,x_n^\pm] \\ \big\downarrow{\scriptstyle \Psi(\cdot,u)} & & \big\downarrow{\scriptstyle \phi_{n,u}} \\ \mathcal{R}_{0p}^{n-1}[u^\pm] & \xrightarrow{\omega_p^{n-1}\times 1} & \mathbb{C}[x_1^\pm,\dots,x_{n-1}^\pm,u^\pm]. \end{array} \tag{2.3.2}$$

By Lemma 2.2.3 in the previous section we can write

$$\alpha_{d,s} = \begin{pmatrix} g^{-1}h & g^{-1}\sigma\tilde{h} \\ 0 & g^T\tilde{h}\end{pmatrix}, \qquad \alpha_{d',s'} = \begin{pmatrix} (g')^{-1}h' & (g')^{-1}\sigma'\tilde{h}' \\ 0 & (g')^T\tilde{h}'\end{pmatrix},$$

where, as in Lemma 2.2.3, (g,h) and σ are coset representatives taken from the sets W_p^n and $(S_p')^n$ respectively.

For any $s\in S_{\mathbb{A}}$ define $\mathbf{g}(s)\in\mathbb{T}$ by

$$\begin{aligned} \gamma(s_p) &:= \int_{\mathbb{Z}_p^n} e_p\left(\frac{x^T s_p x}{2}\right)dx, \\ \mathbf{g}(s) &:= \prod_{p\in\mathbf{f}}\frac{\gamma(s_p)}{|\gamma(s_p)|}. \end{aligned}$$

By Lemma 2.4 of [Shi95b] we then get that $J^{\frac{1}{2}}(\alpha_{d,s},z) = \mathbf{g}(-\sigma)$ and $J^{\frac{1}{2}}(\alpha_{d',s'},z) = \mathbf{g}(-\sigma')$ are both independent of z and therefore

$$J^k(\alpha_{d,s},z) = J(r_P(\alpha_{d,s}))J(r_P(\alpha_{d',s'}))^{-1}p^{a_{d_n}[k]}J^k(\alpha_{d',s'},z). \tag{2.3.3}$$

Remark 2.3.1. In general, the quotient $J(r_P(\alpha_{d,s}))J(r_P(\alpha_{d',s'}))^{-1}$ is not so easy to calculate. By Lemma A1.5 of [Shi00] we can assume $\sigma = \mathrm{diag}[\sigma_1,\dots,\sigma_n]$, where each $\sigma_i\in\mathbb{Q}$. If $\mathrm{ord}_p(\sigma_i)\ge 0$ for all i then $\mathbf{g}(-\sigma) = \mathbf{g}(-\sigma') = 1$ by definition. So let $\mathrm{ord}_p(\sigma_i) = -m_i$ where $0\le m_i\in\mathbb{Z}$ and put $m = \sum_i m_i$. We can further assume that $m_i = 0$ only if $i > e$ for some integer $e\le n$. Then following the proof of Lemma A1.6 of [Shi00] we see that

$$\gamma(-\sigma) = p^{-m}\prod_{i=1}^{e} G(p^{m_i}\sigma_i, p^{m_i}),$$

where $G(a,c) = \sum_{n=1}^{c-1} e^{2\pi i\frac{an^2}{c}}$, for integers a,c. By the assumption $(\mathfrak{b}^{-1},\mathfrak{bc})\subseteq 2\mathbb{Z}\times 2\mathbb{Z}$ we have that $p\ne 2$ since $p\nmid\mathfrak{c}$. Let $\left(\frac{\cdot}{p}\right)$ denote the Legendre symbol and recall for an odd integer b that $\varepsilon_b = 1$ if $b\equiv 1 \pmod 4$ but $\varepsilon_b = i$ if $b\equiv 3 \pmod 4$; note that $\left(\frac{-1}{p}\right) = \varepsilon_p^2$

and $\varepsilon_p^{-1} = \varepsilon_p^2 \varepsilon_p$. Therefore we have

$$G(-p^{m_i}\sigma_i, p^{m_i}) = p^{\frac{m_i}{2}} \varepsilon_{p^{m_i}}^{-1} \left(\frac{-p^{m_i}\sigma_i}{p}\right)^{m_i},$$

which gives

$$\mathbf{g}(-\sigma) = \prod_{i=1}^{e} \varepsilon_{p^{m_i}}^{-1} \left(\frac{-p^{m_i}\sigma_i}{p}\right)^{m_i}.$$

So if $e < n$ then $J(r_P(\alpha_{d,s}))J(r_P(\alpha_{d',s'}))^{-1} = 1$ and if $e = n$ then it is $\varepsilon_{p^{m_n}}^{-1} \left(\frac{-p^{m_n}\sigma_n}{p}\right)^{m_n}$. By multiplying out cosets one can obtain a more concrete expression for the m_i in terms of the $q \in \mathsf{X}_p$ and d appearing in the decomposition of (2.2.7), and therefore of $\Psi(A_q, u)$ itself, but in general this is quite messy.

The above calculation, that the form of $J(r_P(\alpha_{d,s}))$ is a bunch of quadratic characters, directly generalises the cosets α_b^*, β_h^*, and σ^* found in the $n = 1$ case of [Shi73, p. 451]. If $n = 1$ and $q = p \in \mathsf{X}_p^1$ is just our chosen prime number then by multiplying out cosets we have $\mathrm{ord}_p(d) \in \{-1, 0, 1\}$. Using Lemma 2.2.3 for the values of σ we can take, there are the following three cases:

1. $\mathrm{ord}_p(d) = 1$ and we can take $\sigma \in \mathbb{Z}_p/p^2\mathbb{Z}_p$. So $\mathbf{g}(-\sigma) = 1$ and these are the α_b^*.
2. $\mathrm{ord}_p(d) = 0$ and we can take $\sigma \in \mathbb{Z}_p[p^{-1}]$. So $\mathbf{g}(-\sigma)$ is a quadratic character as above and these are the β_h^*.
3. $\mathrm{ord}_p(d) = -1$ and we can take $\sigma = 0$ (in this case $g = 1$ and $h = d$ in the notation of Lemma 2.2.3). So $\mathbf{g}(-\sigma) = 1$ and we obtain the σ^* here.

Proposition 2.3.2. *If $f \in \mathcal{M}_k^n(\Gamma, \psi)$ and $p \nmid \mathfrak{c}$ then*

$$\Phi(f|A_q) = (\Phi f)|\Psi(A_q, \psi_p^{-1}(p)p^{n-[k]}).$$

Proof. Firstly, since $\psi_{\mathfrak{c}}(p) = \psi_p^{-1}(p)$, we have

$$f|A_q = \sum_{x,d,s} \sum_{\tau \in S_+^n} \psi_p^{-1}(|d|) J^k(\alpha_{d,s}, z)^{-1} c_f(\tau, 1) e(d^{-1}\tau\tilde{d}z + \tau s),$$

where x, d, s sum over the same sets as in the decomposition of (2.2.7), and apply Φ to this expression. If $\tau = \left(\begin{smallmatrix} \tau' & \star \\ \star & t \end{smallmatrix}\right)$ with $\tau' \in S_+^{n-1}$ and $0 \leq t \in \mathbb{Z}$, then we know that the last diagonal entry of $d^{-1}\tau\tilde{d}$ is $p^{-2a_{d_n}}t$. Thus writing $z = \left(\begin{smallmatrix} z' & 0 \\ 0 & i\lambda \end{smallmatrix}\right)$ and letting $\lambda \to \infty$, any terms involving τ where $t > 0$ will tend to 0 and we are left only with those $\tau \in S_+^n$ such that $t = 0$ – these are precisely all elements of S_+^{n-1}. For such a $\tau = \left(\begin{smallmatrix} \tau' & 0 \\ 0 & 0 \end{smallmatrix}\right)$ we have

$$d^{-1}\tau\tilde{d}z + \tau s = \begin{pmatrix} (d')^{-1}\tau'\tilde{d}'z' + \tau's' & 0 \\ 0 & 0 \end{pmatrix}.$$

By the relation between $J^k(\alpha_{d,s}, z)$ and $J^k(\alpha_{d',s'}, z)$ of (2.3.3) we get

$$\Phi(f|A_q) = \sum_{x,d,s} J(r_P(\alpha_{d',s'}))J(r_P(\alpha_{d,s}))^{-1}\psi_p^{-1}(p^{a_{d_n}})p^{-a_{d_n}[k]}$$

$$\times \left[\psi_{\mathfrak{c}}(|d'|) J^k(\alpha_{d',s'}, z')^{-1} \sum_{\tau' \in S_+^{n-1}} c_f \left(\begin{pmatrix} \tau' & 0 \\ 0 & 0 \end{pmatrix}, 1 \right) e(\tau'(\tilde{d}' z' + s'd')(d')^{-1}) \right],$$

which is exactly $(\Phi f)|\Psi(A_q, \psi_p^{-1}(p)p^{n-[k]})$, as Φf has Fourier coefficients $c_f\left(\left(\begin{smallmatrix} \tau' & 0 \\ 0 & 0 \end{smallmatrix}\right), 1\right)$ for all $\tau \in S_+^{n-1}$. $\square$

Suppose that f is a non-zero eigenform with eigenvalues given by the homomorphism $\Lambda : \mathcal{R}_0 \to \mathbb{C}$. The above proposition allows a direct comparison of the Satake p-parameters of Φf and f. We saw in the previous section how one obtains these parameters and the whole point of these is to give the action of the Hecke operator $\mathcal{T}_p$ – generally difficult to understand – in terms of a polynomial, (2.2.10), in t acting as a scalar – much easier to understand.

Extend the definitions of Ψ and ω to $\mathcal{T}_p$ by letting them act linearly on the coefficients. For any $1 \le \ell \in \mathbb{Z}$ the polynomial $\omega_p^\ell(x_1, \dots, x_\ell; t) := \omega_p^\ell(\mathcal{T}_p^\ell)$ is given by (2.2.10); if g is an ℓ-degree eigenform with Satake p-parameters $(\lambda_{p,1}, \dots, \lambda_{p,\ell})$ then $g|\mathcal{T}_p^\ell = \omega_p^\ell(\lambda_{p,1}, \dots, \lambda_{p,\ell}; t)g$.

Notice from (2.2.9) that

$$\Phi(f|\mathcal{T}_p^n) = \sum_{m=0}^{\infty} \Lambda(p^m) t^m \Phi f. \tag{2.3.4}$$

Assume $0 \neq \Phi f$ has Satake p-parameters $(\lambda_{p,1}, \dots, \lambda_{p,n-1})$ for $p \nmid \mathfrak{c}$. The aim is to find the Satake p-parameters of f. By the commuting square of (2.3.2) we have

$$\begin{aligned} \omega_p^{n-1}(\Psi(\mathcal{T}_p^n, u)) &= \phi_{n,u}(\omega_p^n(\mathcal{T}_p^n)) \\ &= \left[\prod_{i=1}^{n-1} \frac{1 - p^{2i-1}t^2}{(1 - p^n x_i t)(1 - p^n x_i^{-1} t)} \right] \frac{1 - p^{2n-1}t^2}{(1 - p^n u t)(1 - p^n u^{-1} t)}, \end{aligned}$$

so that

$$(\Phi f)|\Psi(\mathcal{T}_p^n, u) = \left[\prod_{i=1}^{n-1} \frac{1 - p^{2i-1}t^2}{(1 - p^n \lambda_{p,i} t)(1 - p^n \lambda_{p,i}^{-1} t)} \right] \frac{1 - p^{2n-1}t^2}{(1 - p^n u t)(1 - p^n u^{-1} t)} \Phi f. \tag{2.3.5}$$

On the other hand Proposition 2.3.2 along with the identity in (2.3.4) gives

$$(\Phi f)|\Psi(\mathcal{T}_p^n, \psi_p^{-1}(p)p^{n-[k]}) = \Phi(f|\mathcal{T}_p^n) = \sum_{m=0}^{\infty} \Lambda(p^m) t^m \Phi f. \tag{2.3.6}$$

Equating (2.3.5) and (2.3.6) above with $u = \psi_p^{-1}(p)p^{n-[k]}$ proves the following proposition.

Proposition 2.3.3. *Let $f \in \mathcal{M}_k^n(\Gamma, \psi)$ be a non-zero Hecke eigenform such that $\Phi f \neq 0$. Then Φf is an eigenform of degree $n-1$. If Φf has Satake p-parameters $(\lambda_{p,1}, \dots, \lambda_{p,n-1})$ for $p \nmid \mathfrak{c}$ then the Satake p-parameters of f are $(\lambda_{p,1}, \dots, \lambda_{p,n-1}, \psi_p^{-1}(p)p^{n-[k]})$.*

Define the Hecke character $\chi := \psi^{-2}$. Using the above proposition, we can obtain a useful expression between $L_\psi^n(s, f, \chi)$ and $L_\psi^{n-1}(s-1, \Phi f, \chi)$. Assume $\Phi f \neq 0$ has Satake

p-parameters $(\lambda_{p,1}, \ldots, \lambda_{p,n-1})$; by Proposition 2.3.3 above the Euler factor of f at $p \nmid \mathfrak{c}$ is

$$L_p^n((\psi^{\mathfrak{c}}\chi^*)(p)p^{-s}) = L_p^{n-1}((\psi^{\mathfrak{c}}\chi^*)(p)p^{-s+1})(1-\chi^*(p)p^{2n-[k]-s})(1-p^{[k]-s}),$$

and the Euler factors at $p \mid \mathfrak{c}$ are just 1 by definition of χ. Therefore

$$L_\psi^n(s, f, \chi) = L_\psi^{n-1}(s-1, \Phi f, \chi)L(s+[k]-2n, \chi)\zeta_{\mathfrak{c}}(s-[k]),$$

where $\zeta_{\mathfrak{c}}$ is the Riemann zeta function with the Euler factors at $p \mid \mathfrak{c}$ removed. By induction, for any $0 \leq r' \leq n$ such that $\Phi^{n-r'} f \neq 0$, we get

$$\begin{aligned} L_\psi^n(s, f, \chi) = L_\psi^{r'}&(s-n+r', \Phi^{n-r'} f, \chi) \\ &\times \prod_{i=0}^{n-r'-1} L(s+[k]-2n+i, \chi)\zeta_{\mathfrak{c}}(s-[k]-i). \end{aligned} \tag{2.3.7}$$

2.3.2 Klingen Eisenstein series

Let G' denote the image of G under the embedding

$$\begin{aligned} G &\to G_{\mathbb{A}} \\ x &\mapsto (x_v)_v, \end{aligned}$$

where $x_\infty = x$ and $x_p = I_{2n}$ for all primes p, and define

$$\mathfrak{G} := \mathrm{pr}^{-1}(G') \leq M_{\mathbb{A}}.$$

We have that $\mathfrak{G} \leq \mathfrak{M}$. By [Shi95a, p. 544] the group $\mathfrak{G}$ can be identified with the group of couples (α, q), where $\alpha \in G$ and $q : \mathbb{H}_n \to \mathbb{C}$ is a holomorphic function such that $q(z)^2/j(\alpha, z) \in \mathbb{T}$ is a constant, with the group law $(\alpha, q)(\alpha', q') = (\alpha\alpha', q(\alpha' z)q'(z))$ as follows. To each (α, q) there exists a unique $\sigma \in \mathfrak{G}$ such that $\mathrm{pr}(\sigma) = \alpha \in G'$ (i.e. $\mathrm{pr}(\sigma)_\infty = \alpha$ and $\mathrm{pr}(\sigma)_p = I_{2n}$ for all primes p) and $q = h_\sigma$. The group $\mathfrak{G}$ acts on $f : \mathbb{H}_n \to \mathbb{C}$ as

$$(f||_k \xi)(z) = q(z)j(\alpha, z)^{[k]} f(\alpha \cdot z),$$

where $\xi = (\alpha, q) \in \mathfrak{G}$.

Viewing G as a subgroup of $M_{\mathbb{A}}$ as usual through the natural lift $r : G \to M_{\mathbb{A}}$, suppose we have $\alpha \in G \cap \mathfrak{M}$, then the pair (α, h_α) associates to some element $\hat{\alpha}$ of $\mathfrak{G}$ by the identification of the previous paragraph. Therefore we obtain a map $G \cap \mathfrak{M} \to \mathfrak{G}$; $\alpha \mapsto \hat{\alpha}$. A congruence subgroup of $\mathfrak{G}$ is defined as a subgroup Δ such that $\mathrm{pr}(\Delta)$ is a congruence subgroup Γ of G and Δ coincides with the image of Γ under the map $\alpha \mapsto \hat{\alpha}$. As such, congruence subgroups of $G \cap \mathfrak{M}$ and of $\mathfrak{G}$ are one and the same and we shall use the notation Γ to denote congruence subgroups of either G or $\mathfrak{G}$.

For an integer r such that $0 \leq r \leq n$ and for any $\alpha \in M_n(\mathbb{A}_{\mathbb{Q}})$, we write

$$\alpha = \begin{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} & \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \\ \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} & \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \end{pmatrix}, \tag{2.3.8}$$

where, for any $x \in \{a, b, c, d\}$, we have $x_1 \in M_r(\mathbb{A}_{\mathbb{Q}}), x_2 \in M_{r,n-r}(\mathbb{A}_{\mathbb{Q}}), x_3 \in M_{n-r,r}(\mathbb{A}_{\mathbb{Q}})$, and $x_4 \in M_{n-r}(\mathbb{A}_{\mathbb{Q}})$. Also write

$$x_\alpha = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} x_1(\alpha) & x_2(\alpha) \\ x_3(\alpha) & x_4(\alpha) \end{pmatrix}$$

when we wish to emphasise the matrix α to which these blocks belong. If $r = n$ then we understand that $x_\alpha = x_1(\alpha)$ and likewise, for $r = 0$, we have $x_\alpha = x_4(\alpha)$. Define the following parabolic subgroup $P^{n,r} \leq G$ by

$$P^{n,r} := \{\alpha \in G \mid a_2(\alpha) = c_2(\alpha) = 0, c_3(\alpha) = d_3(\alpha) = 0, c_4(\alpha) = 0\} \text{ if } 0 < r < n,$$

with $P^{n,0} := P^n$ and $P^{n,n} := G$. With α of the form in (2.3.8), we have some maps

$$\begin{aligned} \pi_r : M_{2n}(\mathbb{A}_{\mathbb{Q}}) &\to M_{2r}(\mathbb{A}_{\mathbb{Q}}) \\ \alpha &\mapsto \begin{pmatrix} a_1(\alpha) & b_1(\alpha) \\ c_1(\alpha) & d_1(\alpha) \end{pmatrix}, \\ \lambda_r : M_{2n}(\mathbb{A}_{\mathbb{Q}}) &\to \mathbb{A}_{\mathbb{Q}} \\ \alpha &\mapsto |d_4(\alpha)|. \end{aligned}$$

These define respective homomorphisms $P_{\mathbb{A}}^{n,r} \to Sp_r(\mathbb{A}_{\mathbb{Q}})$ and $P_{\mathbb{A}}^{n,r} \to \mathbb{I}_{\mathbb{Q}}$. On the metaplectic side let $\mathfrak{P}^{n,r} := \{(\alpha, q) \in \mathfrak{G} \mid \alpha \in P^{n,r}\}$ and extend π_r and λ_r to $\mathfrak{P}^{n,r}$ by letting

$$\begin{aligned} \pi_r((\alpha, q)) &:= (\pi_r(\alpha), |\lambda_r(\alpha)|^{-\frac{1}{2}} q') \in \mathfrak{G}^r, \\ \lambda_r((\alpha, q)) &:= \lambda_r(a_\infty) \in \mathbb{Q}^\times, \end{aligned}$$

where $q'(z) := q\left(\left(\begin{smallmatrix} z & w \\ w^T & z' \end{smallmatrix}\right)\right)$ does not depend on the choice of w, z'.

Suppose that $0 \leq r \leq n$, $\Gamma \leq \mathfrak{G}^n$ is a congruence subgroup viewed under the previously discussed identification $\gamma \mapsto (\gamma, h_\gamma)$, ψ is a Hecke character in the sense of Definition 2.1.4, and K is some number field, we define for $\mathcal{X} \in \{\mathcal{M}, \mathcal{S}\}$ the space

$$\mathcal{X}_k^r(\Gamma \cap \mathfrak{P}^{n,r}, \psi) := \{f \in \mathcal{X}_k^r \mid f||_k \pi_r(\gamma) = (\operatorname{sgn}^{[k]} \psi_{\mathfrak{c}}^{-1})(\lambda_r(\gamma)) f \text{ for all } \gamma \in \Gamma \cap \mathfrak{P}^{n,r}\},$$

and denote by $\mathcal{X}_k^r(\Gamma \cap \mathfrak{P}^{n,r}, \psi, K)$ the forms in the above space whose Fourier coefficients are all in K. To any $f \in \mathcal{S}_k^r(\Gamma \cap \mathfrak{P}^{n,r}, \psi)$ one can associate the *Klingen Eisenstein series* denoted by $E_k^{n,r}(z, s; f, \psi, \Gamma)$, defined for variables $z \in \mathbb{H}_n, s \in \mathbb{C}$ in [Shi95a, p. 554], and whose sum is convergent for $\mathfrak{Re}(2s) > n+r+1$. Of particular interest to us are the series

$$E_k^{n,r}(z; f, \psi, \Gamma) := \sum_{\gamma \in (\Gamma \cap \mathfrak{P}^{n,r}) \backslash \Gamma} \psi_{\mathfrak{c}}(|a_\gamma|) f\left(z^{(r)}\right) ||_k \gamma,$$

where $z^{(r)}$ is the upper left $r \times r$ block of z, and this latter sum is convergent provided $k > n+r+1$. This was extended to all $k > \frac{n+r+3}{2}$ in [Shi95a, p. 346]. By Lemma 8.11 of [Shi95a] these are holomorphic if $k > 2n$.

Assume that $k > 2n$. For each $0 \le r \le n$, let $\mathrm{Eis}_k^{n,r}$ denote the set of all $E_k^{n,r}(z; f, 1, \Gamma)||_k \alpha$, with α ranging over $\mathfrak{G}^n$, f ranging over $\mathcal{S}_k^r(\Gamma \cap \mathfrak{P}^{n,r})$, and Γ ranging over all congruence subgroups of $\mathfrak{G}^n$ under the usual identification $\gamma \mapsto (\gamma, h_\gamma)$. Then put

$$\mathcal{E}_k^{n,r} := \mathrm{span}_{\mathbb{C}} \mathrm{Eis}_k^{n,r}.$$

As $E_k^{n,n}(z; f, 1, \Gamma) = f$ we have $\mathcal{E}_k^{n,n} = \mathcal{S}_k^n$. Set $\mathcal{E}_k^{n,r}(\Gamma, \psi) = \mathcal{E}_k^{n,r} \cap \mathcal{M}_k^n(\Gamma, \psi)$ for any congruence subgroup Γ. Their eminence in this section comes from a decomposition, in their terms, of the space of modular forms.

Theorem 2.3.4 (Shimura, [Shi95a], pp.581–582)**.** *If $k > 2n$, then we have the decompositions*

$$\mathcal{M}_k^n = \bigoplus_{r=0}^{n} \mathcal{E}_k^{n,r},$$
$$\mathcal{M}_k^n(\Gamma, \psi) = \bigoplus_{r=0}^{n} \mathcal{E}_k^{n,r}(\Gamma, \psi).$$

Remark 2.3.5. Originally in [Shi95a] the above theorem was proven for $k > 2n$, which bound was further improved in [Shi96, p. 346]. We retain that of the former as later results will require $k > 2n$ anyway.

Theorem 27.16 of [Shi00] gives that the above decompositions preserve algebraicity over $\overline{\mathbb{Q}}$.

For any integral ideal $\mathfrak{a}$ let $\mathcal{R}_0^{\mathfrak{a}}$ (resp. $\mathcal{R}_{\mathfrak{a}}(\widehat{\mathfrak{D}}, \widehat{\mathfrak{Z}}_0)$) denote the subspace of $\mathcal{R}_0$ (resp. $\mathcal{R}(\widehat{\mathfrak{D}}, \widehat{\mathfrak{Z}}_0)$) generated by all the A_q (resp. $T_{q,\psi}$) with $q_p \in \mathsf{X}_p$ for all p and $q_p \in \mathsf{O}_p$ for all $p \mid \mathfrak{a}$.

Theorem 2.3.6. *Let $0 \le r' \le r \le n$ be integers, and assume $[k] > \frac{n}{2} + r' + 1$. Consider two non-zero Hecke eigenforms $f \in \mathcal{E}_k^{n,r}(\Gamma, \psi)$ and $f' \in \mathcal{E}_k^{n,r'}(\Gamma, \psi)$ with the same eigenvalues for $\mathcal{R}_0^{\mathfrak{c}}$. Then $r = r'$.*

Proof. As in [Har81, p. 309] we may assume that $r = n$ and therefore that f is a cusp form. Assume for a contradiction that $r' < r$. Since f and f' share the same eigenvalues for $\mathcal{R}_0^{\mathfrak{c}}$, we have $L_\psi^n(s, f, \chi) = L_\psi^n(s, f', \chi)$ where $\chi = \psi^{-2}$ is defined as in the previous subsection. So, using the relation in (2.3.7) obtained at the end of the last subsection, one has

$$L_\psi^n(s, f, \chi) = L_\psi^{r'}(s - n + r', \Phi^{n-r'} f', \chi) \prod_{i=0}^{n-r'-1} L(s + [k] - 2n + i, \chi) \zeta_{\mathfrak{c}}(s - [k] - i).$$

Plug $s = [k] + n - r'$ into this. For $i = n - r' - 1$ we have that $\zeta_{\mathfrak{c}}(s - [k] - i) = \zeta_{\mathfrak{c}}(1)$ is a pole; note $\zeta_{\mathfrak{c}}(s - [k] - i) \neq 0$ for all other i and $L(s + [k] - 2n + i, \chi) \neq 0$ for all i. By Theorem A in [Shi96, p. 332], $L^n_\psi(s, f, \chi)$ is absolutely convergent for $\mathfrak{Re}(s) > \frac{3n}{2} + 1$ and by our choice of s and k we indeed have this. So the left-hand side is *finite*. In the same manner $L^{r'}_\psi(s', \Phi^{n-r'} f', \chi)$ is absolutely convergent for $\mathfrak{Re}(s') > \frac{3r'}{2} + 1$, which inequality $s' = s - n + r'$ satisfies by our choice of s and k. So $L^{r'}_\psi(s - n + r', \Phi^{n-r'} f', \chi)$ is *non-zero*. This gives a contradiction as the right-hand side of this expression contains a pole, yet the left does not. So $r' = r$. $\square$

Definition 2.3.7. For any $0 \leq r \leq n$ we let

$$X_r := \mathfrak{P}^{n,r} \backslash \mathfrak{G}^n / \Gamma$$

be representatives for the r-dimensional cusps.

For notational purposes let

$$\Phi_\xi f := \Phi(f||_k \xi^{-1}),$$

for any $\xi \in X_{n-1}$ and $f \in \mathcal{M}^n_k(\Gamma)$. Then we define the map

$$\begin{aligned} \Phi_\star : \mathcal{M}^n_k(\Gamma, \psi) &\to \prod_{\xi \in X_{n-1}} \mathcal{M}^{n-1}_k(\xi \Gamma \xi^{-1} \cap \mathfrak{P}^{n,n-1}, \psi) \\ f &\mapsto (\Phi_\xi f)_\xi, \end{aligned}$$

and by definition $\ker(\Phi_\star) = \mathcal{S}^n_k(\Gamma, \psi)$.

Lemma 2.3.8. *If $f \in \mathcal{M}^n_k(\Gamma, \psi)$ then $(f||_k \xi^{-1})|A_q = (f|A_q)||_k \xi^{-1}$ for any $\xi \in X_{n-1}$ and any $A_q \in \mathcal{R}^{\mathfrak{c}}_0$.*

Proof. Since ξ is the identity at all finite places we have, for $\sigma = \mathrm{diag}[\tilde{q}, q]$ and $q \in \mathsf{X}$ such that $q_p \in \mathsf{O}_p$ for all $p \mid \mathfrak{c}$, that

$$(\xi D \xi^{-1}) \sigma (\xi D \xi^{-1}) = \xi Sp_n(\mathbb{R}) \xi^{-1} \prod_{p \nmid \mathfrak{c}} D_p \sigma_p D_p,$$

from which $G \cap (\xi D \xi^{-1}) \sigma (\xi D \xi^{-1}) = \xi G \xi^{-1} \cap (D \sigma D) = \xi (\Gamma \beta \Gamma) \xi^{-1}$ for some $\beta \in G \cap Z$. Supposing that $\Gamma \alpha$ are the single cosets in the decomposition of $\Gamma \beta \Gamma$, we have

$$G \cap (\xi D \xi^{-1}) \sigma (\xi D \xi^{-1}) = \xi \left(\bigsqcup_\alpha \Gamma \alpha \right) \xi^{-1} = \bigsqcup_\alpha (\xi \Gamma \xi^{-1})(\xi \alpha \xi^{-1}),$$

so that $\xi \alpha \xi^{-1}$ are the single cosets for the operator A_q of level $\xi \Gamma \xi^{-1}$. Note that $\xi \in \mathfrak{D}[2, 2]$ and that

$$\begin{aligned} (f||_k \xi^{-1})|A_q &= \sum_\alpha j^k(\xi^{-1}, \xi \alpha \xi^{-1} z)^{-1} J^k(\xi \alpha \xi^{-1}, z)^{-1} f(\alpha \xi^{-1} z), \\ (f|A_q)||_k \xi^{-1} &= j^k(\xi^{-1}, z)^{-1} \sum_\alpha J^k(\alpha, \xi^{-1} z)^{-1} f(\alpha \xi^{-1} z). \end{aligned}$$

So all that remains is to show that

$$j^k(\xi^{-1}, \xi\alpha\xi^{-1}z)^{-1} J^k(\xi\alpha\xi^{-1}, z)^{-1} = j^k(\xi^{-1}, z)^{-1} J^k(\alpha, \xi^{-1}z)^{-1},$$

and both sides of this equation are indeed equal to $J^k(\alpha\xi^{-1}, z)^{-1}$ by using properties of J^k found in (2.2.1) and (2.2.2). □

Proposition 2.3.9. *The space $\mathcal{M}_k(\Gamma, \psi)$ has a basis consisting of eigenforms for $\mathcal{R}_0^{\mathfrak{c}}$.*

Proof. Lemma 4.5 of [Shi95b] tells us that $T_{q,\psi}$ (or A_q) are Hermitian on cusp forms provided $q_p \in \mathfrak{O}_p$ for $p \mid \mathfrak{c}$. From this it follows immediately that $\mathcal{S}_k^n(\Gamma, \psi)$ has a basis of eigenforms for $\mathcal{R}_0^{\mathfrak{c}}$.

To show that the Eisenstein series $\mathcal{E}_k^n(\Gamma, \psi)$ has such a basis we use induction on n. Note by [Kob84, p. 210] that the space $\mathcal{M}_k^1(\Gamma, \psi)$ has a basis of eigenforms for $(\mathcal{R}_0^1)^{\mathfrak{c}}$.

We make three claims, which hold for all $1 \leq n \in \mathbb{Z}$:

(1) The space $\mathcal{E}_k^n(\Gamma, \psi)$ is invariant under $(\mathcal{R}_0^n)^{\mathfrak{c}}$.

(2) There exists an epimorphism $(\mathcal{R}_0^n)^{\mathfrak{c}} \to (\mathcal{R}_0^{n-1})^{\mathfrak{c}}; A \mapsto A^*$ such that $\Phi_\xi(f|A) = (\Phi_\xi)|A^*$ for all $\xi \in X_{n-1}$, $f \in \mathcal{M}_k^n(\Gamma)$.

(3) The space $\Phi_\xi \mathcal{E}_k^n(\Gamma, \psi)$ is invariant under $(\mathcal{R}_0^{n-1})^{\mathfrak{c}}$ for all $\xi \in X_{n-1}$.

Claim **(1)** follows from the fact that $\mathcal{S}_k^n(\Gamma, \psi)$ is readily seen to be invariant under $(\mathcal{R}_0^n)^{\mathfrak{c}}$ and by using the self-adjointness of A_q on cusp forms. Claim **(2)** is given by the local maps $\Psi(\cdot, p^{n-[k]}\psi(p))$ of the previous section and Lemma 2.3.8 above. Claim **(3)** follows from the previous two claims; indeed let $A = A_0^*$ for $A \in (\mathcal{R}_0^{n-1})^{\mathfrak{c}}$ and $A_0 \in (\mathcal{R}_0^n)^{\mathfrak{c}}$, then

$$(\Phi_\xi \mathcal{E}_k^n(\Gamma, \psi))|A = \Phi_\xi(\mathcal{E}_k^n(\Gamma, \psi)|A_0) \subseteq \Phi_\xi \mathcal{E}_k^n(\Gamma, \psi).$$

So now assume the proposition holds for $n-1$. By the induction hypothesis we obtain, for each $\xi \in X_{n-1}$, a basis of $\mathcal{M}_k^{n-1}(\xi\Gamma\xi^{-1} \cap \mathfrak{P}^{n,n-1}, \psi)$ consisting of eigenforms. Call this basis $\mathcal{B}_\xi$. Since $\Phi_\xi \mathcal{E}_k^n(\Gamma, \psi) \subseteq \mathcal{M}_k^{n-1}(\xi\Gamma\xi^{-1} \cap \mathfrak{P}^{n,n-1}, \psi)$ is invariant under $\mathcal{R}_0^{\mathfrak{c}}$ then we obtain from $\mathcal{B}_\xi$ a basis, call it $\mathcal{C}_\xi$, of $\Phi_\xi \mathcal{E}_k^n(\Gamma, \psi)$ consisting of eigenforms of $(\mathcal{R}_0^{n-1})^{\mathfrak{c}}$. Let $\mathcal{C}_X$ denote the resultant product basis of $\Phi_\star \mathcal{E}_k^n(\Gamma, \psi)$. As $\ker(\Phi_\star) = \mathcal{S}_k^n(\Gamma, \psi)$ then we have that $\Phi_\star$ is injective on $\mathcal{E}_k^n(\Gamma, \psi)$, and so the inverse image of $\mathcal{C}_X$, call it $\mathcal{C}_\star$, gives a basis for $\mathcal{E}_k^n(\Gamma, \psi)$.

The set $\mathcal{C}_\star$ consists of all $g \in \mathcal{M}_k^n(\Gamma, \psi)$ such that $\Phi_\star(g) \in \mathcal{C}_X$, that is such that $\Phi_{\xi_0}(g)$ is non-zero and belongs to $\mathcal{C}_{\xi_0}$ for some $\xi_0 \in X_{n-1}$ and $\Phi_\xi(g) = 0$ for all other $\xi \neq \xi_0$ – this is by the definition of the product basis. Therefore $f_{\xi_0} := \Phi_{\xi_0}(g) \in \mathcal{C}_{\xi_0}$ is an eigenform, say with eigenvalues Λ_0. For any $A_q \in (\mathcal{R}_0^n)^{\mathfrak{c}}$ we then have

$$\Phi_{\xi_0}(g|A_q) = f_{\xi_0}|A_q^* = \Lambda_0(A_q^*) f_{\xi_0}$$

and we see that $\Phi_\star(g|A_q) = \Lambda_0(A_q^*)\Phi_\star(g)$. By injectivity of $\Phi_\star$, any $g \in \mathcal{C}_\star$ is therefore an eigenform for $(\mathcal{R}_0^n)^{\mathfrak{c}}$. □

Proposition 2.3.10. *Let $V \subseteq \mathcal{M}_k^n(\Gamma,\psi)$ be an eigenspace for $\mathcal{R}_0^{\mathfrak{c}}$ whose eigenvalues are given by the homomorphism $\Lambda : \mathcal{R}_0^{\mathfrak{c}} \to \mathbb{C}$. It is spanned by $V \cap \mathcal{M}_k^n(\mathbb{Q}(\psi,\Lambda))$.*

Proof. Write

$$V = \mathcal{M}_k^n(\Gamma,\psi,\Lambda) := \{f \in \mathcal{M}_k^n(\Gamma,\psi) \mid f|A = \Lambda(A)f \text{ for all } A \in \mathcal{R}_0^{\mathfrak{c}}\}.$$

By the previous proposition the space $\mathcal{M}_k^n(\Gamma,\psi)$ is spanned by eigenforms for $\mathcal{R}_0^{\mathfrak{c}}$ and by Lemma 5.1 in [Bou18] the action of $\mathcal{R}_0^{\mathfrak{c}}$ preserves $\mathcal{M}_k^n(\Gamma,\psi,\mathbb{Q}(\psi,\Lambda))$. As we have a ring of $\mathbb{Q}(\psi,\Lambda)$-linear transformations on $\mathcal{M}_k^n(\Gamma,\psi,\mathbb{Q}(\psi,\Lambda))$ the argument of [Shi00, p. 233] follows. $\square$

So, if $k > 2n$, we obtain an equality of two different direct sum decompositions of $\mathcal{M}_k^n(\Gamma,\psi)$. From Proposition 2.3.9 one of these decompositions consists of eigenspaces for $\mathcal{R}_0^{\mathfrak{c}}$ and by Theorem 2.3.4 the other one consists of the spaces $\mathcal{E}_k^{n,r}(\Gamma,\psi)$. By Theorem 2.3.6 we have that $\mathcal{E}_k^{n,r}(\Gamma,\psi)$ contains entire eigenspaces for $\mathcal{R}_0^{\mathfrak{c}}$. So by the basic properties of direct sums we see that each $\mathcal{E}_k^{n,r}(\Gamma,\psi)$ is itself a direct sum of eigenspaces.

For any character ψ let $\Lambda_{k,\psi} = \Lambda_{k,\psi}^n \subseteq \mathrm{Hom}(\mathcal{R}_0^{\mathfrak{c}},\mathbb{C})$ be the finite subset such that

$$\mathcal{M}_k^n(\Gamma,\psi) = \bigoplus_{\Lambda \in \Lambda_{k,\psi}} \mathcal{M}_k^n(\Gamma,\psi,\Lambda), \tag{2.3.9}$$

and let $\mathbb{Q}(\Lambda_{k,\psi})/\mathbb{Q}$ be the field extension generated by the values of Λ for all $\Lambda \in \Lambda_{k,\psi}$. The above discussion in conjunction with Proposition 2.3.10 proves the following result for $0 \le r \le n-1$; the cusp-form case $r = n$ is already known.

Corollary 2.3.11. *Let $0 \le r \le n$ be integers and assume that $k > 2n$. The space $\mathcal{E}_k^{n,r}(\Gamma,\psi)$ is spanned by $\mathcal{E}_k^{n,r}(\Gamma,\psi,\mathbb{Q}(\psi,\Lambda_{k,\psi})) := \mathcal{E}_k^{n,r}(\Gamma,\psi) \cap \mathcal{M}_k^n(\mathbb{Q}(\psi,\Lambda_{k,\psi}))$.*

We need such an algebraic basis at other cusps as well. Let $\zeta_{\mathfrak{a}} := e^{\frac{2\pi i}{N(\mathfrak{a})}}$ denote the $N(\mathfrak{a})$th root of unity for any integral ideal $\mathfrak{a}$ and recall $\zeta : \mathfrak{M} \to \mathbb{T}$ as the character, see the property in (2.1.1), such that $h(\sigma,z)^2 = \zeta(\sigma)j(\mathrm{pr}(\sigma),z)$. Let

$$\zeta_\star = \zeta|_X, \tag{2.3.10}$$

where $X = \bigcup_r X_r \subseteq \mathfrak{G}^n$ is the set of representatives for all cusps.

Theorem 2.3.12. *Let $K/\mathbb{Q}$ be an algebraic field extension and let $f \in \mathcal{M}_k^n(\Gamma,K)$. For any $0 \le r \le n$ and $\xi \in X_r$ we have $f||_k\xi^{-1} \in \mathcal{M}_k^n(\xi\Gamma\xi^{-1},K(\zeta_{\mathfrak{c}},\zeta_\star))$.*

Proof. In the integral-weight case – $\ell \in \mathbb{Z}$ and $g \in \mathcal{M}_\ell^n(\Gamma,K)$ – Proposition 1.8 of [FC80, p. 146] gives $g||_\ell\xi^{-1} \in \mathcal{M}_\ell(\xi\Gamma\xi^{-1},K(\zeta_{\mathfrak{c}}))$. The half-integral weight case is deduced from this via the use of the theta series $\theta(z) := \sum_{x\in\mathbb{Z}^n} e(\frac{x^Tzx}{2}) \in \mathcal{M}_{\frac{1}{2}}^n(\mathbb{Q})$. By the second equation of Proposition 1.3 [Shi93] the translate $\theta||_{\frac{1}{2}}\xi^{-1} = \theta$ has rational coefficients for any ξ.

Take f as stated in the theorem, the form θf has integral weight $[k+1]$ and coefficients in K. Then we have that $(\theta f)||_{[k+1]}\xi^{-1}$ has coefficients in $K(\zeta_{\mathfrak{c}})$ for any $\xi \in X_r$, and we get

$$\begin{aligned}((\theta f)||_{[k+1]}\xi^{-1})(z) &= |\mu(\xi^{-1}, z)|^{-[k+1]}\theta(\xi^{-1}z)f(\xi^{-1}z)\\ &= j(\xi^{-1}, z)^{-1}h(\xi^{-1}, z)^2(\theta||_{\frac{1}{2}}\xi^{-1})(z)(f||_k\xi^{-1})(z)\\ &= \zeta(\xi^{-1})\theta(z)(f||_k\xi^{-1})(z).\end{aligned}$$

Considering θ as an element of $\mathbb{Q}[[q]]$, with $q = e^{2\pi i}$, it is an invertible power series as it has a non-zero constant term. Considering $\theta^{-1} \in \mathbb{Q}[[q]]$ we then have

$$f||_k\xi^{-1} = \zeta(\xi)\theta^{-1}(\theta f)||_{[k+1]}\xi^{-1} \in \mathcal{M}_k(\xi\Gamma\xi^{-1}, K(\zeta_{\mathfrak{c}}, \zeta_\star)).$$

□

Remark 2.3.13. For certain congruence subgroups one can remove the $\zeta_\star$. For example, if Γ has cusps only at 0 and ∞ then $X = \{I_{2n}, \iota\}$. In this case $\zeta(\iota) = (-i)^n$ by Proposition 1.1R of [Shi93] and we see that $\mathbb{Q}(\zeta_\star) \subseteq \mathbb{Q}(\zeta_{\mathfrak{c}})$ since $4 \mid \mathfrak{c}$. In general, however, it is not so clear that we can remove $\zeta_\star$; the relative non-triviality of the behaviour of f at the cusps in the half-integral weight setting (see Proposition 1.4 of [Shi93] and subsequent discussion) suggests that it may be a necessary addition.

Corollary 2.3.14. *Let $0 \leq r \leq n$ be integers and assume $k > 2n$. The space $\mathcal{E}_k^{n,r}(\xi\Gamma\xi^{-1}, \psi)$ is spanned by $\mathcal{E}_k^{n,r}(\xi\Gamma\xi^{-1}, \psi, \mathbb{Q}(\Lambda_{k,\psi}, G(\psi), \zeta_\star))$.*

Proof. As $\mathcal{E}_k^{n,r}(\xi\Gamma\xi^{-1}, \psi) = \mathcal{E}_k^{n,r}(\Gamma, \psi)||_k\xi^{-1}$ this follows from Corollary 2.3.11 and Theorem 2.3.12 above. □

The previously defined map $\Phi_\star$ provides a useful isomorphism from which we can determine the rationality of $\Phi_\star f$ given that of f.

Theorem 2.3.15 (Shimura, [Shi95a], p. 582; [Shi96], p. 347)**.** *Let $k > 2n$ and fix $r < n$. The map*

$$\Phi_\star^{n-r} : \mathcal{E}_k^{n,r}(\Gamma, \psi) \to \prod_{\xi \in X_r} \mathcal{S}_k^r(\xi\Gamma\xi^{-1} \cap \mathfrak{P}^{n,r}, \psi)$$

is a $\mathbb{C}$-linear isomorphism.

Corollary 2.3.16. *If $f \in \mathcal{E}_k^{n,r}(\Gamma, \psi)$ and $k > 2n$, then $f \in \mathcal{M}_k(\Gamma, \psi, \mathbb{Q}(\Lambda_{k,\psi}, G(\psi), \zeta_\star))$ if and only if*

$$\Phi_\star^{n-r} f \in \prod_{\xi \in X_r} \mathcal{S}_k^r(\xi\Gamma\xi^{-1} \cap \mathfrak{P}^{n,r}, \psi, \mathbb{Q}(\Lambda_{k,\psi}, G(\psi), \zeta_\star)).$$

Proof. Theorem 2.3.12 and the fact that $\Phi^{n-r}(\mathcal{M}_k^n(\Gamma, \psi, L)) \subseteq \mathcal{M}_k^r(\Gamma, \psi, L)$ for any subfield $L \subseteq \mathbb{C}$ gives necessity.

For sufficiency, let $\{g_1^n, \ldots, g_m^n\}$ be a basis of $\mathbb{Q}(\Lambda_{k,\psi}, G(\psi), \zeta_\star)$)-rational forms for $\mathcal{E}_k^{n,r}(\Gamma, \psi)$. By Corollary 2.3.14 there also exists a basis of $\mathbb{Q}(\Lambda_{k,\psi}, G(\psi), \zeta_\star)$)-rational forms for each $\mathcal{S}_k^r(\xi\Gamma\xi^{-1} \cap \mathfrak{P}^{n,r}, \psi)$, so let $\{g_1^r, \ldots, g_m^r\}$ denote the product basis for $\prod_\xi \mathcal{S}_k^r(\xi\Gamma\xi^{-1} \cap \mathfrak{P}^{n,r}, \psi)$

obtained out of this. Each g_i^r is zero for all $\xi \in X_r$ except for one element whereby it belongs to $\mathcal{S}_k^r(\xi\Gamma\xi^{-1} \cap \mathfrak{P}^{n,r}, \psi, \mathbb{Q}(\Lambda_{k,\psi}, G(\psi), \zeta_\star))$. Assume further that this basis is ordered to that $\Phi_\star^{n-r}(g_i^n) = g_i^r$ for all i – we can do this by the isomorphism in Theorem 2.3.15. Writing $f = \sum_{i=1}^m \alpha_i g_i^n \in \mathcal{E}_k^{n,r}(\Gamma, \psi)$, the proof follows from the following argument that each $\alpha_i \in \mathbb{Q}(\Lambda_{k,\psi}, G(\psi), \zeta_\star)$. By assumption

$$\Phi_\star^{n-r} f = \sum_{i=1}^m \alpha_i g_i^r \in \prod_{\xi \in X_r} \mathcal{S}_k^r(\xi\Gamma\xi^{-1} \cap \mathfrak{P}^{n,r}, \psi, \mathbb{Q}(\Lambda_{k,\psi}, G(\psi), \zeta_\star)).$$

If $m = 1$ and $f \neq 0$ then by Lemma 8.2 (2) and Lemma 8.11 (4) of [Shi95a] there exists $\xi \in X_r$ such that $\Phi^{n-r}(f||_k\xi^{-1}) \neq 0$. As $\alpha_1 = (g_1^r)^{-1}\Phi_\star^{n-r} f$ we see by assumption that $\alpha_1 \in \mathbb{Q}(\Lambda_{k,\psi}, G(\psi), \zeta_\star)$. The rest follows by induction on m. $\square$

All of the above results allow us to now prove a particular case of Garrett's conjecture in Theorem 2.3.19 below.

Lemma 2.3.17 (Shimura, [Shi95a], p. 578)**.** *If* $f \in \mathcal{S}_k^r(\xi\Gamma\xi^{-1} \cap \mathfrak{P}^{n,r}, \psi), \xi \in X_r$, *and* $k > n + r + 1$ *then we have*

$$\Phi^{n-r}[E_k^{n,r}(z; f, \psi, \xi\Gamma\xi^{-1})||_k \xi\nu^{-1}] = \begin{cases} f & \text{if } \nu = \xi \\ 0 & \text{if } \nu \in X_r \text{ and } \nu \neq \xi. \end{cases}$$

Remark 2.3.18. The above lemma is given in [Shi95a] with trivial character and the proof found there follows directly from Lemma 8.5 of that paper. That lemma clearly applies for non-trivial character Klingen Eisenstein series as well, hence the above formulation.

For any $f \in \mathcal{E}_k^{n,r}(\Gamma, \psi)$ and any $0 \leq r \leq n$ define

$$F_k^{n,r}(z; f, \psi, \Gamma) := \sum_{\xi \in X_r} E_k^{n,r}(z; \Phi_\xi^{n-r} f, \psi, \xi\Gamma\xi^{-1})||_k \xi \in \mathcal{E}_k^{n,r}(\Gamma, \psi).$$

Theorem 2.3.19. *If* $0 \leq r \leq n$, $k > n+r+1$, *and* $f \in \mathcal{E}_k^{n,r}(\Gamma, \psi, \mathbb{Q}(\Lambda_{k,\psi}, G(\psi), \zeta_\star))$, *then* $F_k^{n,r}(z; f, \psi, \Gamma) \in \mathcal{E}_k^{n,r}(\Gamma, \psi, \mathbb{Q}(\Lambda_{k,\psi}, G(\psi), \zeta_\star))$.

Proof. The case $r = n$ is given immediately by Theorem 2.3.12, so assume $r < n$. For any $\nu \in X_r$ we have $\Phi_\nu^{n-r} F_k^{n,r}(z; f, \psi, \Gamma) = \Phi_\nu^{n-r} f$ by Lemma 2.3.17. By Theorem 2.3.12, $\Phi_\nu^{n-r} f$ has coefficients in $\mathbb{Q}(\Lambda_{k,\psi}, G(\psi), \zeta_\star)$ for each $\nu \in X_r$. If $0 \leq r \leq n-1$ then, by Corollary 2.3.16, $F_k^{n,r}(z; f, \psi, \Gamma)$ also has coefficients in $\mathbb{Q}(\Lambda_{k,\psi}, G(\psi), \zeta_\star)$. $\square$

Define the space of all Eisenstein series by

$$\mathcal{E}_k^n := \prod_{r=0}^{n-1} \mathcal{E}_k^{n,r}.$$

The decomposition $\mathcal{M}_k^n = \mathcal{S}_k^n \oplus \mathcal{E}_k^n$ of Theorem 2.3.4 is proven inductively and each step involves the use of the Eisenstein series $F_k^{n,r}$. Observing this proof along with Theorem 2.3.19 gives the following.

Theorem A. *Assume that $k > 2n$. We have*

$$\mathcal{M}_k(\Gamma, \psi, \mathbb{Q}(\Lambda_{k,\psi}, G(\psi), \zeta_\star)) = \mathcal{S}_k(\Gamma, \psi, \mathbb{Q}(\Lambda_{k,\psi}, G(\psi), \zeta_\star)) \oplus \mathcal{E}_k(\Gamma, \psi, \mathbb{Q}(\Lambda_{k,\psi}, G(\psi), \zeta_\star)).$$

Proof. The proof of Theorem 2.3.4 is outlined, from [Shi95a, pp. 581–582], as follows. If $f \in \mathcal{M}_k(\Gamma, \psi)$ then put $f_0 = F_k^{n,0}(z; f, \psi, \Gamma)$. By Lemma 2.3.17 we have $\Phi_\star^n(f - f_0) = 0$, so $\Phi_\nu^{n-1}(f - f_0)$ is a cusp form for any ν. Then put $f_1 = F_k^{n,1}(z; f - f_1, \psi, \Gamma)$ and repeat the above procedure to get $f_2 = F_k^{n,2}(z; f - f_0 - f_1, \psi, \Gamma)$ and so on until f_n which is a cusp form. In the final step we obtain $0 = \Phi^0(f - f_0 - \cdots - f_n) = f - f_0 - \cdots - f_n$ and this is the proof of Theorem 2.3.4. So if f has coefficients in $\mathbb{Q}(\Lambda_{k,\psi}, G(\psi), \zeta_\star)$ then, by Theorem 2.3.19, so do each of $f_0, f_1, \ldots, f_n$. □

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2.4 The Rankin-Selberg method

Notation. k – half-integral weight; $[k] = k - \frac{1}{2} \in \mathbb{Z}$.
$\mathfrak{b}$ – fractional ideal of $\mathbb{Q}$; $\mathfrak{c}$ – integral ideal of $\mathbb{Z}$.
$(\mathfrak{b}^{-1}, \mathfrak{bc}) \subseteq 2\mathbb{Z} \times 2\mathbb{Z}$.
$\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}]$.
$\Gamma_n(s) = \pi^{\frac{n(n-1)}{4}} \prod_{i=0}^{n-1} \Gamma(s - \frac{i}{2})$.
$\Delta(z) = |\,\mathfrak{Im}(z)|$ for $z \in \mathbb{H}_n$.
$S_+ = \{\xi \in M_n(\mathbb{Q}) \mid \xi^T = \xi, \xi \geq 0\}$; $\tau \in S_+$.
$X = \{x \in M_n(\mathbb{R}) \mid x^T = x, -\frac{1}{2} \leq x_{ij} \leq \frac{1}{2}\}$.
$Y = \{y \in M_n(\mathbb{R}) \mid y^T = y, y > 0\}$.
ρ_τ – quadratic character associated to $\mathbb{Q}(i^{[n/2]}\sqrt{|2\tau|})/\mathbb{Q}$.
$\mathsf{X}_p = M_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p)$; $\mathsf{O}_p = GL_n(\mathbb{Z}_p)$;
$\mathsf{X} = GL_n(\mathbb{Q})_\mathbf{f} \cap \prod_p \mathsf{X}_p$; $\mathsf{O} = \prod_p GL_n(\mathbb{Z}_p)$.

The Rankin-Selberg method is a powerful tool in the theory of automorphic forms and can be used to obtain a variety of results, an example of which is Theorem 2.2.6 concerning the analyticity of L-functions. In Chapters 3, 4, and 5 of this thesis respectively it is critically used to determine algebraicity of special L-values for metaplectic modular forms, prove existence of metaplectic p-adic L-functions, and give analyticity and non-vanishing results of L-functions for vector-valued modular forms. The method, whose objective is to express an L-function as an integral of certain automorphic forms multiplied by an Eisenstein series, was developed independently by Rankin in [Ran39] and Selberg in [Sel40]. They used it to prove analytic continuation and a functional equation of the L-function of the type

$$D(s, f, g) := \sum_{n=1}^{\infty} c_f(n, 1)\overline{c_g(n, 1)}n^{-s},$$

where $f, g \in \mathcal{M}_\ell^1$ and $\ell \in \mathbb{Z}$.

The application of the method to Siegel modular forms of integral weight has a substantial history, it is used for example by Sturm to prove algebraicity of L-values in [Stu81]. Traditionally, the integral expression was slightly limited in that certain Euler factors were removed from the L-function – see Proposition 2.8 of [Pan91, p. 57] in which the Euler factors dividing $N(\mathfrak{c})|2\tau|$ are removed. In his paper [Shi94], Shimura removes this limitation entirely in the integral-weight case and this was achieved by the study of a certain type of Dirichlet series α and its factorisation into certain polynomials – see Chapter 3 and Theorem 3.2 in particular of [Shi94]. In his further paper, [Shi95b], Shimura replicates these results for the half-integral weight case and subsequently gives an integral expression of full generality in [Shi96, (4.1)]. In this section then, we restate this expression and give its derivation in suitable detail following closely the proofs and results of Shimura mentioned supra; it is therefore entirely expository and the results contained within are well known.

2.4.1 Definitions and the integral expression

Let $f \in \mathcal{S}_k(\Gamma, \psi)$ be a non-zero eigenform and take a normalised Hecke character ψ, defined in Definition 2.1.4, satisfying the usual properties of (2.1.5) and (2.1.6). Fix $\tau \in S_+$, where recall $S_+ = \{\xi = \xi^T \in M_n(\mathbb{Q}) \mid \xi \geq 0\}$, such that the Fourier coefficient of f at τ satisfies $c_f(\tau, 1) \neq 0$. Let χ be a normalised Hecke character, in the sense of Definition 2.1.4, with conductor $\mathfrak{f}$, and take $\mu \in \{0, 1\}$ such that $(\psi\chi)_\infty(x) = \operatorname{sgn}(x_\infty)^{[k]+\mu}$.

For any $1 \leq m \in \mathbb{Z}$, $\kappa \in \frac{1}{2}\mathbb{Z}$, and integral ideal $\mathfrak{a}$, we define certain products of Dirichlet L-functions as follows:

$$L_{\mathfrak{a}}(s, \eta) = \prod_{p \nmid \mathfrak{a}} (1 - \eta^*(p)p^{-s})^{-1},$$

$$\Lambda_{\mathfrak{a}}^{m,\kappa}(s, \eta) = \begin{cases} L_{\mathfrak{a}}(2s, \eta) \displaystyle\prod_{i=1}^{[\frac{m}{2}]} L_{\mathfrak{a}}(4s - 2i, \eta^2) & \text{if } \kappa \in \mathbb{Z}, \\ \displaystyle\prod_{i=1}^{[\frac{m+1}{2}]} L_{\mathfrak{a}}(4s - 2i + 1, \eta^2) & \text{if } \kappa \notin \mathbb{Z}. \end{cases} \tag{2.4.1}$$

Then, for a congruence subgroup $\Gamma_0 = \Gamma[\mathfrak{x}^{-1}, \mathfrak{x}\mathfrak{y}]$ (contained in $\mathfrak{M}$ if $\kappa \notin \mathbb{Z}$) and normalised Hecke character $\eta : \mathbb{I}_{\mathbb{Q}}/\mathbb{Q}^\times \to \mathbb{T}$ satisfying (2.1.5) (with $\mathfrak{y}$ in place of $\mathfrak{c}$) and (2.1.7), we normalise the Eisenstein series of (2.1.8) by

$$\mathcal{E}_\kappa(z, s) = \mathcal{E}_\kappa(z, s; \eta, \Gamma_0) := \overline{\Lambda_{\mathfrak{y}}^{n,\kappa}(s, \psi)} E_\kappa(z, \bar{s}; \eta, \Gamma_0). \tag{2.4.2}$$

Let $\mathbf{b}$ be the set of primes p such that $p \nmid \mathfrak{c}$ and $\operatorname{ord}_p(|\tau|) \neq 0$. For each $p \in \mathbf{b}$ there exists a certain polynomial $g_p \in \mathbb{Z}[t]$ such that $g_p(0) = 1$. These polynomials are the $f_\zeta(t)$ in Proposition 4.1 of [Shi95b]; the g_p arise as the integral factor of a certain Dirichlet series α – which can be seen in the second line of (2.4.15) below or see Theorem 5.2 of [Shi95b] for more details – and it is exactly this factorisation that allows Shimura to overcome the traditional limitations of this method in removing Euler factors.

Recall $\mathfrak{t}$ as an integral ideal such that $h^T(2\tau)h \in 4\mathfrak{t}^{-1}$ for all $h \in \mathbb{Z}^n$. Now take the ideals $\mathfrak{x} = \mathfrak{b}$ and $\mathfrak{y} = \mathfrak{c} \cap (4\mathfrak{t}\mathfrak{f}^2)$ in the subgroup Γ_0, and put

$$\Lambda_{\mathfrak{a}}(s) = \Lambda_{\mathfrak{a}}^{n,k-n/2-\mu}(s,\eta), \tag{2.4.3}$$

for any integral ideal $\mathfrak{a}$. Let $\eta := \psi\chi\rho_\tau$, where ρ_τ is the quadratic character defined just after the definition of the theta series in (2.1.9). The character η satisfies (2.1.7), with $\kappa = k - \frac{n}{2} - \mu$, by choice of μ. Finally, recall the definition of the generalised Gamma function Γ_n from (2.1.13) and the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{y}}$ from (2.1.14).

From (4.1) of [Shi96], the desired integral expression is given as

$$\begin{aligned} L_\psi(s,f,\chi) = {} & \left[\Gamma_n\left(\tfrac{s-n-1+k+\mu}{2}\right) 2c_f(\tau,1)\right]^{-1} N(\mathfrak{b})^{\frac{n(n+1)}{2}} |4\pi\tau|^{\frac{s-n-1+k+\mu}{2}} \left(\tfrac{\Lambda_{\mathfrak{c}}}{\Lambda_{\mathfrak{y}}}\right)\left(\tfrac{2s-n}{4}\right) \\ & \times \prod_{p\in\mathbf{b}} g_p\left((\psi^{\mathfrak{c}}\chi^*)(p)p^{-s}\right) \langle f, \theta_\chi \mathcal{E}_{k-\frac{n}{2}-\mu}(\cdot, \tfrac{2s-n}{4}; \bar{\eta}, \Gamma_0)\rangle_{\mathfrak{y}} \operatorname{Vol}(\Gamma_0\backslash\mathbb{H}_n). \end{aligned} \tag{2.4.4}$$

2.4.2 Unfolding

The derivation of the above integral expression comes in two main steps. First, in this section, we give an integral expression for the Rankin-Selberg Dirichlet series $D(s,f,g)$ of two modular forms f and g. This is achieved by the *unfolding* procedure which makes use of the identity

$$\int_{\Gamma\backslash\mathbb{H}_n} \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \varphi(\gamma\cdot z)d^\times z = \int_{P\cap\Gamma\backslash\mathbb{H}_n} \varphi(z)d^\times z, \tag{2.4.5}$$

where $\varphi : \mathbb{H}_n \to \mathbb{C}$ is a $P\cap\Gamma$-invariant function. The second step, to come in the next subsection, establishes an expression relating $D(s,f,g)$ to $L(s,f,\chi)$ when $g = \theta_\chi$ and f is a non-zero eigenform.

For now, let $f \in \mathcal{S}_k(\Gamma,\psi)$ and $g \in \mathcal{M}_\ell(\Gamma',\psi')$ where $\ell \le k$ is any integral or half-integral weight; $\Gamma' = \Gamma[(\mathfrak{b}')^{-1}, \mathfrak{b}'\mathfrak{c}']$, with $((\mathfrak{b}')^{-1}, \mathfrak{b}'\mathfrak{c}') \subseteq 2\mathbb{Z}\times 2\mathbb{Z}$ if $\ell \notin \mathbb{Z}$; and ψ' is a Hecke character satisfying the usual property of (2.1.5) with $\mathfrak{c} = \mathfrak{c}'$, with the further property that $(\psi/\psi')_\infty(x) = \operatorname{sgn}(x_\infty)^{[k-\ell]}$.

By choice of ψ' the Hecke character $\eta := \psi(\psi')^{-1}$ satisfies (2.1.7) with $\kappa = k-\ell$. Fix $\mathfrak{b}$ and $\mathfrak{b}'$, let $\mathfrak{x} := \mathfrak{b}\cap\mathfrak{b}'$, $\mathfrak{y} := \mathfrak{x}^{-1}(\mathfrak{b}\mathfrak{c}\cap\mathfrak{b}'\mathfrak{c}')$, and $\Gamma_0 := \Gamma[\mathfrak{x}^{-1},\mathfrak{x}\mathfrak{y}]$. We start by looking at the integral $\langle f, gE_{k-\ell}\rangle_{\mathfrak{y}}$, in which

$$E_{k-\ell}(z) := E_{k-\ell}\left(z, \bar{s} + \tfrac{n+1}{2}; \bar{\eta}, \Gamma_0\right).$$

Using the definition, (2.1.8), of the Eisenstein series this integral transforms to look like the left-hand side of (2.4.5), allowing the application of the unfolding procedure. To see how this works exactly, the integral $\mathrm{Vol}(\Gamma_0\backslash\mathbb{H}_n)\langle f, gE_{k-\ell}\rangle_{\mathfrak{y}}$ is equal to

$$\begin{aligned}&\int_{\Gamma_0\backslash\mathbb{H}_n}\sum_{\gamma\in P\cap\Gamma_0\backslash\Gamma_0}\psi_{\mathfrak{c}}(|a_\gamma|)(f||_k\gamma^{-1})(\gamma z)\overline{\psi'_{\mathfrak{c}'}(|a_\gamma|)(g||_\ell\gamma^{-1})(\gamma z)}\Delta(\gamma z)^{s+\frac{n+1+k+\ell}{2}}d^\times z\\ &=\int_{\Gamma_0\backslash\mathbb{H}_n}\sum_{\gamma\in P\cap\Gamma_0\backslash\Gamma_0}f(\gamma z)\overline{g(\gamma z)}\Delta(\gamma z)^{s+\frac{n+1+k+\ell}{2}}d^\times z,\end{aligned}\tag{2.4.6}$$

where we used $\psi_p(|a_\gamma|)=1=\psi'_q(|a_\gamma|)$ if $p\nmid\mathfrak{c}$ and $q\nmid\mathfrak{c}'$; that $\psi_{\mathfrak{c}}(|a_\gamma|)f||_k\gamma^{-1}=f$ and $\psi'_{\mathfrak{c}'}(|a_\gamma|)g||_\ell\gamma^{-1}=g$ for any $\gamma\in\Gamma_0$; and the following facts

$$\begin{aligned}j_\gamma^{k-\ell}(z)^{-1}&=j_\gamma^k(z)^{-1}j_\gamma^\ell(z)=j_{\gamma^{-1}}^k(\gamma z)j_{\gamma^{-1}}^\ell(\gamma z)^{-1},\\ \overline{j_{\gamma^{-1}}^k(\gamma z)}&=j_{\gamma^{-1}}^k(\gamma z)^{-1}\|\mu(\gamma,z)\|^{-2k},\\ \Delta(\gamma\cdot z)^k&=\|\mu(\gamma,z)\|^{-2k}\Delta(z)^k.\end{aligned}$$

It is easy to check that $\varphi(z):=f(z)\overline{g(z)}\Delta(z)^{s+\frac{n+1+k+\ell}{2}}$ is $P\cap\Gamma_0$-invariant, so we apply the unfolding procedure of (2.4.5) to the above integral (2.4.6) and use the Fourier expansions of f and g to obtain

$$\sum_{\sigma_i\in S_+}\int_{X'}e^{2\pi i\,\mathrm{tr}((\sigma_1-\sigma_2)x)}dx\int_{Y'}c_f(\sigma_1,1)\overline{c_g(\sigma_2,1)}|y|^{s+\frac{k+\ell}{2}}e^{-2\pi\,\mathrm{tr}((\sigma_1+\sigma_2)y)}d^\times y,\tag{2.4.7}$$

where $Y':=Y/SL_n(\mathbb{Z})$ is defined with respect to $y\mapsto aya^T$ with $a\in SL_n(\mathbb{Z})$ and $y\in Y$ (see (2.1.12) for the definition of Y), and $X'\times Y'$ is a domain for $P\cap\Gamma_0\backslash\mathbb{H}_n$. If $P_0:=P\cap\Gamma[1,1]$ then we can take the domain $X\times Y'$ for $P_0\backslash\mathbb{H}_n$, where X is defined in (2.1.11). Note that P_0 acts as the identity on $Y/SL_n(\mathbb{Z})$ and as translation on X', with both integrands in (2.4.7) above being P_0-invariant. The map $P\cap\Gamma_0\backslash P_0\to S(\mathbb{Z}/\mathfrak{x}^{-1})$ given by $p\mapsto b_p^Td_p$ is an isomorphism. So the integral over X' becomes $N(\mathfrak{x}^{-1})^{\frac{n(n+1)}{2}}$ many integrals of $e^{2\pi i\,\mathrm{tr}((\sigma_1-\sigma_2)x)}$ over X, which latter integral is non-zero only when $\sigma_1=\sigma_2$ at which point it is equal 1. Hence the integral $\mathrm{Vol}(\Gamma_0\backslash\mathbb{H}_n)\langle f,gE_{k-\ell}\rangle_{\mathfrak{y}}$, from (2.4.7) above, becomes

$$N(\mathfrak{x})^{-\frac{n(n+1)}{2}}\int_{Y'}\sum_{\sigma\in S_+}c_f(\sigma,1)\overline{c_g(\sigma,1)}|y|^{s+\frac{k+\ell}{2}}e^{-4\pi\,\mathrm{tr}(\sigma y)}d^\times y\tag{2.4.8}$$

We say any $\sigma_1\sim\sigma_2$ in S_+ if $\sigma_1=a^T\sigma_2a$ for some $a\in GL_n(\mathbb{Z})$ and let $S_+/GL_n(\mathbb{Z})$ denote the representatives of this equivalence. Let $U_\sigma:=\{a\in GL_n(\mathbb{Z})\mid a^T\sigma a=\sigma\}$ and set $\nu_\sigma:=\#U_\sigma$. We get

$$\begin{aligned}&\sum_{\sigma\in S_+}c_f(\sigma,1)\overline{c_g(\sigma,1)}e^{-4\pi\,\mathrm{tr}(\sigma y)}\\ &\qquad=2\sum_{a\in SL_n(\mathbb{Z})}\sum_{\sigma\in S_+/GL_n(\mathbb{Z})}\nu_\sigma^{-1}c_f(a^T\sigma a,1)\overline{c_g(a^T\sigma a,1)}e^{-4\pi\,\mathrm{tr}(\sigma aya^T)}.\end{aligned}\tag{2.4.9}$$

The factor of 2 accounts for the action of $-I_n\in GL_n(\mathbb{Z})$. By (iii) and (iv) of Theorem 2.1.5 we have $c_f(a^T\sigma a,1)\overline{c_g(a^T\sigma a,1)}=\eta_\infty(|a|)|a|^{[k]-[\ell]}c_f(\sigma,1)\overline{c_g(\sigma,1)}=c_f(\sigma,1)\overline{c_g(\sigma,1)}$ for any $a\in GL_n(\mathbb{Z})$. Hence the following definition is well defined.

Definition 2.4.1. Let $f\in\mathcal{M}_k(\Gamma,\psi)$ and $g\in\mathcal{M}_\ell(\Gamma',\psi')$ as above. For a complex variable

s we define the *Rankin-Selberg Dirichlet series of f and g* by

$$D(s,f,g) := \sum_{\sigma \in S_+/GL_n(\mathbb{Z})} \nu_\sigma^{-1} c_f(\sigma,1)\overline{c_g(\sigma,1)}|\sigma|^{-s-\frac{k+\ell}{2}}.$$

Now, using (2.4.9) with (2.4.8), we have that $\mathrm{Vol}(\Gamma_0\backslash\mathbb{H}_n)\langle f, gE_{k-\ell}\rangle_{\mathfrak{y}}$ becomes

$$2N(\mathfrak{x})^{-\frac{n(n+1)}{2}} \sum_{\sigma \in S_+/GL_n(\mathbb{Z})} \nu_\sigma^{-1} c_f(\sigma,1)\overline{c_g(\sigma,1)} \int_{Y'} \sum_{a \in SL_n(\mathbb{Z})} |y|^{s+\frac{k+\ell}{2}} e^{-4\pi \operatorname{tr}(\sigma a y a^T)} d^\times y.$$

By the definition of $Y' = Y/SL_n(\mathbb{Z})$ (specifically that it is given with respect to the action $y \mapsto aya^T$), as well as the fact that $|aya^T| = |y|$, we have

$$\int_{Y'} \sum_{a \in SL_n(\mathbb{Z})} |y|^{s+\frac{k+\ell}{2}} e^{-4\pi \operatorname{tr}(\sigma a y a^T)} d^\times y = (4\pi)^{-n(s+\frac{k+\ell}{2})}\Gamma_n(s+\tfrac{k+\ell}{2})|\sigma|^{-s-\frac{k+\ell}{2}},$$

and therefore the culmination of this section is the expression

$$\mathrm{Vol}(\Gamma_0\backslash\mathbb{H}_n)\langle f, gE_{k-\ell}\rangle_{\mathfrak{y}} = 2(4\pi)^{-n(s+\frac{k+\ell}{2})} N(\mathfrak{x})^{-\frac{n(n+1)}{2}} \Gamma_n\left(s+\tfrac{k+\ell}{2}\right) D(s,f,g). \tag{2.4.10}$$

2.4.3 The Dirichlet series of f

Definition 2.4.2. Let $f \in \mathcal{S}_k(\Gamma, \psi)$ be a non-zero eigenform, let $\tau \in S_+$, and let χ be a Hecke character. We define the *Dirichlet series of f at τ* for a complex variable $s \in \mathbb{C}$ by

$$D_\tau(s, f, \chi) := \sum_{x \in \mathsf{X}/\mathsf{O}} (\psi\chi^*)(|x|) c_f(\tau, x) |x|^{-s} \|x\|_{\mathbb{A}}^{-n-1}.$$

The above Dirichlet series can be used to relate $L_\psi(s, f, \chi)$ to the Rankin-Selberg Dirichlet series $D(s, f, \theta_\chi)$ from the previous subsection. Therefore the main integral expression, (2.4.4), will follow from the integral expression, (2.4.10), of $D(s, f, \theta_\chi)$ above.

Now take $\tau \in S_+$ such that $c_f(\tau, 1) \neq 0$. Define a Dirichlet series $\mathcal{T}_{\psi,\chi}$ with coefficients in $\mathcal{R}_0$ by

$$\mathcal{T}_{\psi,\chi} := \sum_{n=1}^{\infty} A(n)(\psi^{\mathfrak{c}}\chi^*)(n) n^{-s}, \tag{2.4.11}$$

where $A(n)$ is the sum of A_q for all $q \in \mathsf{O}\backslash\mathsf{X}/\mathsf{O}$ such that $|q| = n$, and the action of this on f, denoted $f|\mathcal{T}_{\psi,\chi}$, is given by acting coefficient-wise. Now let $\Lambda(n) := \Lambda(A(n))$ be the eigenvalues of f and put $c_{\mathcal{T}}(\tau, x) := c(\tau, x; f|\mathcal{T}_{\psi,\chi})$; immediately we have

$$c_{\mathcal{T}}(\tau, 1) = \left(\sum_{n=1}^{\infty} \Lambda(n)(\psi^{\mathfrak{c}}\chi^*)(n) n^{-s} \right) c_f(\tau, 1). \tag{2.4.12}$$

On the other hand one can use the definition of Hecke operators and the coset decompositions given in Lemma 2.2.3 to obtain

$$c_{\mathcal{T}}(\tau, 1) = \alpha_{\mathfrak{c}}(\tau) D_\tau(s, f, \chi), \tag{2.4.13}$$

where, for any $\zeta \in S_{\mathbf{f}}$ such that $e_{\mathbf{f}}(\operatorname{tr}(S_{\mathbf{f}}(\mathbb{Z})\zeta)) = 1$, we define $\alpha_{\mathfrak{c}}$ by $\alpha_{\mathfrak{c}}(\zeta) := \prod_{p \nmid \mathfrak{c}} \alpha_p(\zeta_p)$ and

$$\alpha_p(\zeta_p) := \sum_{\sigma = d^{-1}c \in S_p/S(\mathbb{Z}_p)} (\psi^{\mathfrak{c}}\chi^*)(|d|) \mathbf{g}(\sigma) e_p(-\operatorname{tr}(\zeta_p \sigma)) |d|^{-s}; \tag{2.4.14}$$

in the above sum we have decomposed $\sigma = d^{-1}c$ into its numerator $c \in M_n(\mathbb{Z}_p)$ and denominator $d \in M_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p)$, which satisfy $c\mathbb{Z}_p + d\mathbb{Z}_p = \mathbb{Z}_p$. This is Theorem 5.1 of [Shi95b].

By Definition 2.2.5, Equation (2.2.11) of this thesis, and the identity of [Shi95b, (5.5)], we also have

$$\begin{gathered} \Lambda_{\mathfrak{c}}^{2n,k}\left(\tfrac{s}{2}, \psi\chi\right) \sum_{n=1}^{\infty} \Lambda(n)(\psi^{\mathfrak{c}}\chi^*)(n) n^{-s} = L_\psi(s, f, \chi), \\ \Lambda_{\mathfrak{c}}^{n,k-\frac{n}{2}-\mu}\left(\tfrac{s}{2}, \psi\chi\rho_\tau\right) \alpha_{\mathfrak{c}}(\tau) = \prod_{p \in \mathbf{b}} g_p((\psi^{\mathfrak{c}}\chi^*)(p) p^{-s}), \end{gathered} \tag{2.4.15}$$

where $\mu \in \{0, 1\}$, and both $\mathbf{b}$ and $g_p \in \mathbb{Q}[t]$ are as in Section 2.4.1.

Combining (2.4.12), (2.4.13), and (2.4.15) above relates $D_\tau(s, f, \chi)$ and $L_\psi(s, f, \chi)$ as follows

$$c_f(\tau,1)L_\psi(s,f,\chi) = \prod_{p\in\mathbf{b}} g_p((\psi^{\mathfrak{c}}\chi^*)(p)p^{-s})\Lambda_{\mathfrak{c}}\left(\tfrac{2s-n}{4}\right)D_\tau(s,f,\chi), \tag{2.4.16}$$

where, recall, $\Lambda_{\mathfrak{c}}(s) = \Lambda_{\mathfrak{c}}^{n,k-\frac{n}{2}-\mu}(s,\psi\chi\rho_\tau)$.

We now relate $D_\tau(s,f,\chi)$ with $D(s,f,\theta_\chi)$. We have $GL_n(\mathbb{A}_\mathbb{Q}) = GL_n(\mathbb{Q})GL_n(\mathbb{R})\mathsf{O}$ by strong approximation. Equation (3.7) of [Shi96] then tells us that

$$D_\tau(s,f,\chi) = \sum_{x\in\mathsf{X}'/\mathsf{O}'}(\psi_{\mathbf{f}}\chi^*)(|x|)c_f(\tau,x)|x|^{n+1-s},$$

where $\mathsf{X}' := M_n(\mathbb{Z})\cap GL_n(\mathbb{Q})$ and $\mathsf{O}' := GL_n(\mathbb{Z})$. Using Theorem 2.1.5 (iii) we have

$$D_\tau(s,f,\chi) = \sum_{x\in\mathsf{X}'/\mathsf{O}'}(\chi_\infty\chi^*)(|x|)c_f(x^T\tau x,1)|x|^{n+1-k-s}, \tag{2.4.17}$$

where, since x is global and modulo $GL_n(\mathbb{Z})$, we have been able to use that $(\psi\chi)_\infty(|x|) = 1$ and that $\psi(|x|) = 1$.

Assuming $\chi_\infty(x) = \operatorname{sgn}(x_\infty)^{n\mu}$, the Fourier coefficients of θ_χ are given, by the definition (2.1.9), by

$$c_\theta(\sigma,1) = \sum_{x\in\mathsf{X}'_\sigma}(\chi_\infty\chi^*)^{-1}(|x|)|x|^\mu,$$

where $\sigma\in S_+$ and

$$\mathsf{X}'_\sigma := \begin{cases}\{x\in\mathsf{X}' \mid \sigma = x^T\tau x\} & \text{if } \mathfrak{f}\neq\mathbb{Z},\\ \{x\in M_n(\mathbb{Z}) \mid \sigma = x^T\tau x\} & \text{if } \mathfrak{f}=\mathbb{Z}.\end{cases} \tag{2.4.18}$$

So, since there is support only when $\sigma = x^T\tau x$, we have

$$\begin{aligned} D(s,f,\theta_\chi) &= \sum_{\sigma\in S_+/GL_n(\mathbb{Z})}\nu_\sigma^{-1}\sum_{x\in\mathsf{X}'_\sigma}(\chi_\infty\chi^*)(|x|)c_f(x^T\tau x,1)|x|^\mu|x^T\tau x|^{-s-\frac{k+n/2+\mu}{2}}\\ &= |\tau|^{-s-\frac{k+n/2+\mu}{2}}\sum_{x\in\mathsf{X}'/\mathsf{O}'}(\chi_\infty\chi^*)(|x|)c_f(x^T\tau x,1)|x|^{-2s-k-\frac{n}{2}}. \end{aligned} \tag{2.4.19}$$

Equations (2.4.17) and (2.4.19) then give

$$D_\tau(s,f,\chi) = |\tau|^{\frac{s-n-1+k+\mu}{2}}D\left(\tfrac{2s-3n-2}{4},f,\theta_\chi\right). \tag{2.4.20}$$

Now take $\mu\in\{0,1\}$ such that $(\psi\chi)_\infty(|x|) = \operatorname{sgn}(|x|)^{[k]+\mu}$, therefore $\eta = \psi\chi\rho_\tau$ satisfies (2.1.7) and the integral expression of (2.4.10) holds for $D(s,f,\theta_\chi)$. The Rankin-Selberg integral expression (2.4.4) is subsequently given by combining the three key equalities (2.4.10), (2.4.16), and (2.4.20), recalling that the second argument of $E_{k-\ell}$ is $\bar{s}+\frac{n+1}{2}$ in the expression of (2.4.10).

Chapter 3

Algebraicity of metaplectic L-functions

This chapter consists of results that can also be found in [Mer18b].

In rudimentary terms, special values of L-functions are a set of values for which the transcendental factor can be precisely determined and for which one is able to say something about the algebraic remnant. As was mentioned in the introduction, such notions are generalisations of Euler's calculation that $\zeta(2) \in \pi^2\mathbb{Q}$ and that $\pi^{-2n}\zeta(2n) \in \mathbb{Q}$ for any $1 \leq n \in \mathbb{Z}$.

The above idea has natural extensions to more general L-functions, such as those associated to modular forms, and in the introduction we saw how these can often be interpreted algebraically. In such a case one can therefore approach the proof of these special values either algebraically or analytically. For Siegel eigenforms f of integral weight k, even degree n, and character ψ, Sturm in [Stu81] used the Rankin-Selberg method to prove the special values of the L-function associated to f; the key insight of his method was the development of a holomorphic projection operator **Pr**, which maps certain elements of $C_k^\infty(\Gamma,\psi)$ to $\mathcal{M}_k(\Gamma,\psi)$ and was defined by its explicit action on the Fourier coefficients. Through this method Sturm showed, [Stu81, pp. 347–348], that

$$\pi^{-nk+\frac{3n^2-2n}{4}}c^+(f)L_\psi(m,f,\chi) \in \mathbb{Q}(f,\psi,\chi), \tag{3.0.1}$$

for all $m \in \Omega(\chi)$, where $\Omega(\chi)$ is a set of integers and $c^+(f) \in \mathbb{C}$ (given explicitly in [Stu81]) is known as the period of f.

The work of Sturm was limited by the following aspects of the non-holomorphic Eisenstein series involved in the Rankin-Selberg integral: **(1)** the need for the Eisenstein series to be of integral weight (i.e. the need that n be even); **(2)** lack of algebraicity of the Fourier coefficients of the Eisenstein series beyond the range of convergence for s (limiting the lower bound for $\Omega(\chi)$); **(3)** the need for the projection map to produce a holomorphic *cusp form* for which the Eisenstein series had to be of "bounded growth" (limiting both the upper and lower bound of $\Omega(\chi)$). Since that paper, the theory of Eisenstein series of integral or half-integral weight was substantially developed by Shimura, see for example Chapters 16 and 17 of [Shi00]. Subsequently for $k \in \frac{1}{2}\mathbb{Z}$ he gave, in [Shi00, Theorem 28.8], the special values

$$\pi^a \langle f, f\rangle^{-1} L(m, f, \chi) \in \overline{\mathbb{Q}}$$

for a larger set $m \in \Omega$ (containing $\Omega(\chi)$ if $k \in \mathbb{Z}$) and an integer a depending on n, k, and m.

In this chapter we extend the method of Sturm to metaplectic L-functions – the standard L-functions we associate to a metaplectic eigenform, see Definition 2.2.5 – and use up-to-date theory of Eisenstein series to circumvent the limitations **(1)** – **(3)** detailed above, showing precise algebraicity akin to (3.0.1) for all the values $m \in \Omega$. To deal with **(1)** and **(2)** we use the work of Bouganis, [Bou18, Theorem 3.2], which builds upon the Eisenstein series theory of Shimura mentioned above, and the results of Section 2.3 allow us to remove the limitation of **(3)**. Similar results were proved by alternative means by Bouganis – Theorem 6.2 of [Bou18]; the use of this particular method allows us to remove the condition that $\mu \neq 0$ appearing on the Hecke character found there. Note that the work of Bouganis and Shimura in the above citations work over general totally real fields, whereas we restrict to $\mathbb{Q}$.

We begin this chapter by stating the main theorems. In Section 3.2 we define Sturm's holomorphic projection operator in this setting and prove the existence of its desired properties. Certain cases when this projection is applicable to the non-holomorphic Eisenstein series are given in Section 3.3 and we use this in Section 3.4 to modify the Rankin-Selberg integral expression by applying the projection. The chapter is then concluded by the proof of the main theorems in Section 3.5.

3.1 Main theorems

Notation. k – half-integral weight; $[k] = k - \frac{1}{2} \in \mathbb{Z}$.
$\mathfrak{b}$ – fractional ideal of $\mathbb{Q}$; $\mathfrak{c}$ – integral ideal of $\mathbb{Z}$.
$(\mathfrak{b}^{-1}, \mathfrak{bc}) \subseteq 2\mathbb{Z} \times 2\mathbb{Z}$.
$\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}]$.
Hecke character $\psi : \mathbb{I}_{\mathbb{Q}}/\mathbb{Q}^{\times} \to \mathbb{T}$ satisfying (2.1.5) and (2.1.6) with $\kappa = k$.
$S_+ = \{\xi \in M_n(\mathbb{Q}) \mid \xi^T = \xi, \xi \geq 0\}$; $\tau \in S_+$.
$S^{\triangledown}$ – set of symmetric half-integral $n \times n$ matrices.
ρ_τ – quadratic character associated to $\mathbb{Q}(i^{[n/2]}\sqrt{|2\tau|})/\mathbb{Q}$.
$\delta = n \pmod 2 \in \{0, 1\}$.

The non-holomorphic Eisenstein series $\mathcal{E}_\kappa(z, s; \eta, \Gamma_0)$ defined by (2.1.8) and (2.4.2), plays a key role in the proofs of the main theorems of this chapter. Its non-holomorphicity needs to be addressed with holomorphic projection and, with it, the algebraicity of its Fourier coefficients; this section begins with the latter.

Let $\delta := n \pmod 2 \in \{0,1\}$. For any $\ell \in \frac{1}{2}\mathbb{Z}$ define the set

$$\Omega_0 := \left\{ s \in \tfrac{1}{2}\mathbb{Z} \middle| \left| s - \tfrac{n+1}{4} \right| + \tfrac{n+1}{4} - \tfrac{k-\ell}{2} \in \mathbb{Z}, \tfrac{n+1-k+\ell}{2} \leq s \leq \tfrac{k-\ell}{2} \right\}. \tag{3.1.1}$$

There are exceptional cases where the Eisenstein series $\mathcal{E}_\kappa(z, \frac{2m-n}{4}; \eta, \Gamma_0)$ has different behaviour. The relevant ones are:

$$m = n+1 \text{ and } \eta^2 = 1; \tag{$\mathbf{X}$}$$

$$n = 1, m = \tfrac{3}{2}, \text{ and } \eta = 1; \tag{$\mathbf{R1}$}$$

$$n > 1, m = n + \tfrac{3}{2}, \text{ and } \eta^2 = 1. \tag{$\mathbf{R2}$}$$

Case (**X**) affects neither of the main results, Theorem B1 and B2, since the set of special values are strict half-integers. Neither Case (**R1**) nor (**R2**) affect the first result, Theorem B1, since the set of special values excludes them, but they will have an effect on the second result, Theorem B2.

The following theorem, which is a particular case of Theorem 3.2 of [Bou18], gives the existence of a period $\omega(\eta)$ through which algebraicity of the Fourier coefficients of the Eisenstein series is given.

Theorem 3.1.1 (Bouganis, [Bou18], Theorem 3.2)**.** *Let $\ell \in \frac{1}{2}\mathbb{Z}$ satisfy $k - \ell > \frac{n+1}{2}$. Put $\Gamma_0 = \Gamma[\mathfrak{x}^{-1}, \mathfrak{x}\mathfrak{y}]$ (contained in $\mathfrak{M}$ if $k - \ell \notin \mathbb{Z}$). Let η be a normalised Hecke character defined by Definition 2.1.4 satisfying the usual property of (2.1.5), with $\mathfrak{y}$ in place of $\mathfrak{c}$, as well as (2.1.7) with $\kappa = k - \ell$. Exclude case (**X**). For any m such that $\frac{2m-n}{4} \in \Omega_0$ we have*

$$\mathcal{E}_{k-\ell}(z, \tfrac{2m-n}{4}; \eta, \Gamma_0) = |\pi y|^{-r} \sum_{\substack{\tau \in S_+ \\ \tau \in N(\mathfrak{x})S^\triangledown}} P(\tau, \eta, y) e(\mathrm{tr}(\tau z)),$$

*where $r = \frac{k-\ell}{2} - \frac{2m-n}{4} + 1$ in cases (**R1**) and (**R2**), otherwise $r = \frac{k-\ell}{2} - |\frac{2m-2n-1}{4}| - \frac{n+1}{4}$, and $P(\tau, \eta, y) \in \mathbb{Q}_{ab}[\pi y_{ij} \mid 1 \leq i \leq j \leq n]$. Set*

$$\beta_m := \tfrac{n}{2}(k - \ell + m - n) + \tfrac{\delta}{4},$$

and define a period $\omega_\ell(m, \eta) = \omega_\ell(\eta) = \omega(\eta)$ by

$$\omega(\eta) := \begin{cases} i^{n\left|\frac{2k-2\ell-2m+n}{4}\right| + mn - \frac{3n^2-1}{4}} G(\eta) G(\eta^{n-1}) & \text{if } k - \ell \in \mathbb{Z} \text{ and } m > n, \\ i^{n\left|\frac{2k-2\ell-3n-2+2m}{4}\right| - \frac{n}{2}(3n+2-2m)} G(\eta)^n & \text{if } k - \ell \in \mathbb{Z} \text{ and } m \leq n, \\ i^{n\left|\frac{2k-2\ell-2m+n}{4}\right| - n(k-\ell)} G(\eta\zeta)^n \zeta_8 & \text{if } k - \ell \notin \mathbb{Z}, n \in 2\mathbb{Z},\ m > n, \\ i^{n\left|\frac{2k-2\ell-2m+n}{4}\right| - n(k-\ell) + \nu} (2i)^{\frac{3n}{2} - m} G(\eta\zeta)^n G(\eta) \zeta_8 & \text{if } k - \ell \notin \mathbb{Z}, n \notin 2\mathbb{Z}, m > n, \\ i^{n\left|\frac{2k-2\ell-3n-2+2m}{4}\right| - \frac{n}{2}(3n+2-2m)} G(\eta\zeta)^n \zeta_8 & \text{if } k - \ell \notin \mathbb{Z} \text{ and } m \leq n, \end{cases}$$

where ζ_8 is a fixed 8th root of unity, ζ is the character induced by $h_\gamma(z)^2 = \zeta(\gamma) j(\gamma, z)$ of (2.1.1), and $\nu = 1$ if $n \equiv 1 \pmod 4$, but $\nu = 0$ otherwise. Then we have

$$\left[\frac{P(\tau, \eta, y)}{\pi^{\beta_m} \omega(\eta)} \right]^\sigma = \frac{P(\tau, \eta^\sigma, y)}{\pi^{\beta_m} \omega(\eta^\sigma)},$$

for any $\sigma \in \mathrm{Gal}(\mathbb{Q}_{ab}/\mathbb{Q})$ acting on $P(\tau, \eta, y)$ coefficients-wise.

If $\ell \in \frac{n}{2} + \mathbb{Z}$, then notice that $k - \ell \in \mathbb{Z}$ if and only if n is odd. In such a case relabel

$$\omega_\delta(\eta) := \omega_\ell(\eta).$$

Given a homomorphism $\Lambda : \mathcal{R}_0 \to \mathbb{C}$ and a non-zero eigenform $f \in \mathcal{S}_k(\Gamma, \psi)$ with eigenvalues given by Λ, we take $\varepsilon \in \{0, 1\}$ such that

$$\psi_\infty(x) = \mathrm{sgn}(x_\infty)^{[k]+\varepsilon}.$$

By (2.1.6) we have $\varepsilon = 0$ if n is odd. The period $\omega_\delta(\eta)$ for the Eisenstein series will form part of the overall period of our L-function along with a non-zero complex constant $\mu'(\Lambda, k, \psi)$, dependent only on Λ, k, and ψ, defined by

$$\mu'(\Lambda, k, \psi) := 2^{\frac{\delta}{2}} \pi^{-a} i^{-\frac{n^2}{2}} \omega_\delta(m_\varepsilon, \overline{\psi})^{-1} L_\psi(m_\varepsilon, f), \tag{3.1.2}$$

where $m_\varepsilon := k - 2n - 2 - \varepsilon$, $a := \beta_{m_\varepsilon} + n(k-r) - \frac{n^2+\delta}{4}$, $\ell = \frac{n}{2} + \varepsilon$, and $L_\psi(s, f) := L_\psi(s, f, 1)$. This constant is non-zero for $k > \frac{7n}{2} + 3 + \varepsilon$ as $L_\psi(s, f)$ is absolutely convergent for $\mathfrak{Re}(s) > \frac{3n}{2} + 1$, see [Shi96, Theorem A].

Theorem B1. *Assume that $k > \frac{7n}{2} + 3$ if n is odd, $k > \frac{7n}{2} + 3 + \varepsilon$ if n is even. Let $f \in \mathcal{S}_k(\Gamma, \psi)$ be a non-zero eigenform with eigenvalues Λ. Take a Hecke character χ, choose $\mu \in \{0, 1\}$ such that $(\chi\psi)_\infty(x) = \mathrm{sgn}(x_\infty)^{[k]+\mu}$, put $\eta = \psi\chi\rho_\tau$, and define*

$$\Omega'_{n,k} := \left\{ m \in \tfrac{1}{2}\mathbb{Z} \,\middle|\, \tfrac{k-m-\mu}{2} \in \mathbb{Z},\ \tfrac{3n}{2} + 1 < m < k - 2n - \mu \right\}.$$

For any $\tau \in S_+$ such that $c_f(\tau, 1) \neq 0$ and any $m \in \Omega'_{n,k}$, normalise the L-function by

$$Y_\psi(m, f, \chi) := |\tau|^{\frac{\delta}{2}} \pi^{-b_m} \mu'(\Lambda, k, \psi)^{-1} \omega_\delta(m, \bar{\eta})^{-1} L_\psi(m, f, \chi),$$

where $b_m := \beta_m + n(k-r) - \frac{n^2+\delta}{4}$ and $\ell = \frac{n}{2} + \mu$. Then for any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ we have $Y_\psi(m, f, \chi)^\sigma = Y_{\psi^\sigma}(m, f^\sigma, \chi^\sigma)$ and therefore

$$Y_\psi(m, f, \chi) \in \mathbb{Q}(f, \psi, \chi).$$

The results of Section 2.3 will enable an improvement of the above theorem in two ways. The bound on k is improved and the full range of special values seen, for example, in Theorem 28.8 of [Shi00] is obtained. The periods and powers of π here are slightly different; for $m \in \frac{1}{2}\mathbb{Z}$, $\ell = \frac{n}{2} + \mu$, and $\mu \in \{0, 1\}$, define

$$c_m := \beta_m + n(k-r) - \tfrac{n^2-4n+\delta}{4}$$

in cases (**R1**) and (**R2**), and otherwise let

$$c_m := \begin{cases} \beta_m + n(k-r) - \frac{n^2+\delta}{4} & \text{if } m > n, \\ \beta_m + n(k+m-r) - \frac{5n^2+2n+\delta}{4} & \text{if } m \leq n. \end{cases} \tag{3.1.3}$$

Define a slightly different constant $\mu(\Lambda, k, \psi)$ by

$$\mu(\Lambda, k, \psi) := 2^{\frac{\delta}{2}} \pi^{-c_{k-\varepsilon}} i^{-\frac{n^2}{2}} \omega_\delta(k-\varepsilon, \overline{\psi})^{-1} L_\psi(k-\varepsilon, f). \tag{3.1.4}$$

This constant is non-zero for $k > \frac{3n}{2} + 1 + \varepsilon$.

Theorem B2. *Assume that* $k > \max\{2n, \frac{3n}{2}+1\}$ *if* n *is odd,* $k > \max\{2n, \frac{3n}{2}+1+\varepsilon\}$ *if* n *is even, and let* $f \in \mathcal{S}_k(\Gamma, \psi)$ *be a non-zero eigenform with eigenvalues given by* Λ. *Take a Hecke character* χ, *choose* $\mu \in \{0,1\}$ *such that* $(\chi\psi)_\infty(x) = \operatorname{sgn}(x_\infty)^{[k]+\mu}$, *put* $\eta = \psi\chi\rho_\tau$, *and define*

$$\begin{aligned}
\Omega^+_{n,k} &:= \left\{ m \in \tfrac{1}{2}\mathbb{Z} \,\middle|\, \tfrac{k-m-\mu}{2} \in \mathbb{Z}, n < m \leq k-\mu \right\}, \\
\Omega^-_{n,k} &:= \left\{ m \in \tfrac{1}{2}\mathbb{Z} \,\middle|\, \tfrac{m+k-\mu-1}{2} \in \mathbb{Z}, 2n+1-k+\mu \leq m \leq n \right\}, \\
\Omega_{n,k} &:= \Omega^-_{n,k} \cup \Omega^+_{n,k}.
\end{aligned}$$

For any $\tau \in S_+$ *such that* $c_f(\tau, 1) \neq 0$ *and any* $m \in \Omega_{n,k}$, *normalise the L-function by*

$$Z_\psi(m, f, \chi) := |\tau|^{\frac{\delta}{2}} \pi^{-c_m} \omega_\delta(m, \bar{\eta})^{-1} \mu(\Lambda, k, \psi)^{-1} L_\psi(m, f, \chi).$$

Then $Z_\psi(m, f, \chi)^\sigma = Z_{\psi^\sigma}(m, f^\sigma, \chi^\sigma)$ *for any* $\sigma \in \operatorname{Aut}(\mathbb{C}/\mathbb{Q}(\Lambda_{k,\psi}, G(\psi), \zeta_\star))$ *where, recall,* $\Lambda_{k,\psi}$ *is defined by the decomposition of (2.3.9) and* $\zeta_\star$ *is, as in (2.3.10), the restriction of the character* ζ *to the cusps of* $\Gamma\backslash\mathbb{H}_n$. *Therefore*

$$Z_\psi(m, f, \chi) \in \mathbb{Q}(f, \chi, \Lambda_{k,\psi}, G(\psi), \zeta_\star).$$

Remark 3.1.2. In all cases we actually have $b_m = c_m = n(k+m-n)$ – note that $\ell = \frac{n}{2} + \mu$ in β_m and r; they are therefore integers, and agree with the powers of π present in Theorem 28.8 of [Shi00]. We present the powers of π as a sum of its constituents in order to clarify the proof of these main theorems throughout this chapter.

3.2 Holomorphic projection

Notation. k – half-integral weight; $[k] = k - \frac{1}{2} \in \mathbb{Z}$.
$\mathfrak{b}$ – fractional ideal of $\mathbb{Q}$; $\mathfrak{c}$ – integral ideal of $\mathbb{Z}$.
$(\mathfrak{b}^{-1}, \mathfrak{bc}) \subseteq 2\mathbb{Z} \times 2\mathbb{Z}$.
$\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}]$.
Hecke character $\psi : \mathbb{I}_\mathbb{Q}/\mathbb{Q}^\times \to \mathbb{T}$ satisfying (2.1.5) and (2.1.6) with $\kappa = k$.
$\Delta(z) = |\,\mathfrak{Im}(z)|$ for $z \in \mathbb{H}_n$.
$\mu(\gamma, z) = c_\gamma z + d_\gamma$ for $\gamma \in G$ and $z \in \mathbb{H}_n$.
$\delta = n \pmod 2 \in \{0, 1\}$.
$S^\triangledown$ – set of symmetric half-integral $n \times n$ matrices.
$S^\triangledown_\mathfrak{b}$ – set of symmetric $n \times n$ matrices such that $\tau \in N(\mathfrak{b})S^\triangledown$.

The holomorphic projection given in this section directly generalises Sturm's operator of

Theorem 1 of [Stu81] and Theorem 4.2 of [Pan91, p. 71]. It is applicable, in two cases, to $F \in C_k^\infty(\Gamma, \psi)$ satisfying certain growth conditions.

Definition 3.2.1. If $F \in C_k^\infty(\Gamma, \psi)$ then we say it is of *bounded growth* if, for all $\varepsilon > 0$, we have

$$\int_X \int_Y |F(z)| \Delta(z)^{k-1-n} e^{-\varepsilon \operatorname{tr}(y)} dy dx < \infty, \tag{3.2.1}$$

where X and Y are the sets of symmetric matrices defined in (2.1.11) and (2.1.12) respectively. We say that F is of *moderate growth* if, for all $z \in \mathbb{H}_n$ and sufficiently large $\mathfrak{Re}(s) \gg 0$, the integral

$$\int_{\mathbb{H}_n} F(w) \det(\bar{w} - z)^{-k} |\det(\bar{w} - z)|^{-2s} \Delta(w)^{k+s} d^\times w \tag{3.2.2}$$

is absolutely convergent and can be continued analytically over all $s \in \mathbb{C}$ to the point $s = 0$.

If F has bounded growth then holomorphic projection will yield a holomorphic cusp form, whereas if F has only moderate growth then it will yield a holomorphic modular form, all the while preserving the Petersson inner product. Recall from Section 2.1 that $F \in C_k^\infty(\Gamma, \psi)$ has an absolutely convergent Fourier expansion of the following form

$$F(z) = \sum_{\tau \in S_{\mathfrak{b}}^\triangledown} c_F(\tau, y) e(\operatorname{tr}(\tau x)),$$

where $S_{\mathfrak{b}}^\triangledown$ is the set of all $\tau \in S$ such that $\tau \in N(\mathfrak{b}) S^\triangledown$, and $c_F(\tau, y)$ are smooth functions of y having values in $\mathbb{C}$.

Theorem 3.2.2. *Assume that $k > 2n$ and let $F \in C_k^\infty(\Gamma, \psi)$ have either moderate or bounded growth. For any $0 < \tau \in S_{\mathfrak{b}}^\triangledown$ define*

$$\begin{aligned} c(\tau) &:= \mu(k,n)^{-1} |4\tau|^{k - \frac{n+1}{2}} \int_Y c_F(\tau, y) e^{-2\pi \operatorname{tr}(\tau y)} |y|^{k-1-n} dy, \\ \mu(k,n) &:= \Gamma_n\left(k - \tfrac{n+1}{2}\right) \pi^{-n(k - \frac{n+1}{2})}. \end{aligned}$$

Then the holomorphic projection map is given by

$$\begin{aligned} \mathbf{Pr} : C_k^\infty(\Gamma, \psi) &\to \mathcal{X}_k(\Gamma, \psi) \\ F &\mapsto \sum_{0 < \tau \in S_{\mathfrak{b}}^\triangledown} c(\tau) e(\operatorname{tr}(\tau z)), \end{aligned}$$

in which $\mathcal{X} = \mathcal{S}$ if F is of bounded growth, whereas $\mathcal{X} = \mathcal{M}$ if F is of moderate growth. Furthermore we have $\langle F, g \rangle = \langle \mathbf{Pr}(F), g \rangle$ for any $g \in \mathcal{S}_k(\Gamma', \psi)$ and $\Gamma' \leq \Gamma$ of finite index.

The proof of the above theorem rests on the use of Poincaré series and well-known properties of theirs. One uses a single-variable series to prove the case where F is of bounded growth, and a double-variable Poincaré series when F is of moderate growth. To accommodate half-integral weights, the main change to the integral-weight proof occurs in the definition of these series by including the factor of automorphy h_σ. With this change in the definition, it is clear that all the other relevant properties of the Poincaré series hold, and thus the above theorem. We detail this in the case where F has bounded growth now and give a brief outline for the case where F has moderate growth.

Assume that $k > 2n$, fix a $0 < \tau \in S_{\mathfrak{b}}^{\triangledown}$, and let Γ_∞ be the subgroup of Γ generated by $\left(\begin{smallmatrix} \pm I_n & b \\ 0 & \pm I_n \end{smallmatrix}\right)$, with $b \in M_n(\mathfrak{b}^{-1})$. The single-variable holomorphic Poincaré series is

$$G_\tau(z) := \sum_{\alpha \in \Gamma_\infty \backslash \Gamma} \psi_{\mathfrak{c}}^{-1}(|a_\alpha|) j_\alpha^k(z)^{-1} e(\mathrm{tr}(\tau\alpha z)).$$

Proposition 3.2.3.

(i) The sum defining G_τ converges absolutely and uniformly on compact subgroups of $\mathbb{H}_n$ and $G_\tau \in \mathcal{S}_k(\Gamma, \psi)$.

(ii) If $F \in C_k^\infty(\Gamma, \psi)$ then

$$N(\mathfrak{b})^{\frac{n(n+1)}{2}} \mathrm{Vol}(\Gamma\backslash\mathbb{H}_n)\langle F, G_\tau\rangle = \int_Y c_F(\tau, y) e^{-2\pi\,\mathrm{tr}(\tau y)} |y|^{k-1-n} dy,$$

which integral is absolutely convergent.

(iii) If $f \in \mathcal{S}_k(\Gamma, \psi)$ then

$$N(\mathfrak{b})^{\frac{n(n+1)}{2}} \mathrm{Vol}(\Gamma\backslash\mathbb{H}_n)\langle f, G_\tau\rangle = c_f(\tau, 1)|4\tau|^{\frac{n+1}{2}-k}\mu(k, n).$$

Proof. **(i)** The absolute and uniform convergence, modularity, and cuspidality of G_τ was proven in the integral-weight case by Godement in [God58]. By using the bound $|\mu(\alpha, z)|^{k'} < |\mu(\alpha, z)^k| < |\mu(\alpha, z)|^{k''}$, where $k' = [k]$ and $k'' = [k] + 1$ if $|\mu(\alpha, z)| > 1$, but $k' = [k] + 1$ and $k'' = [k]$ otherwise, we deduce the convergence and cuspidality in the half-integral weight case from the integral-weight case.

To show modularity, we prove $G_\tau||_k\gamma = \psi_{\mathfrak{c}}(|a_\gamma|)G_\tau$ for any $\gamma \in \Gamma$. By the automorphic property of h_σ, see (2.1.3), we have $h(\alpha\gamma, z) = h(\alpha, \gamma z)h(\gamma, z)$, and the usual cocycle relation gives $\mu(\alpha\gamma, z) = \mu(\alpha, \gamma z)\mu(\gamma, z)$; hence we have

$$j_\alpha^k(\gamma z) = j_\gamma^k(z)^{-1} j_{\alpha\gamma}^k(z)$$

and, moreover, $\psi_{\mathfrak{c}}(|a_\alpha|) = \psi_{\mathfrak{c}}(|a_{\alpha\gamma}|)\psi_{\mathfrak{c}}^{-1}(|a_\gamma|)$ since $a_{\alpha\gamma} \equiv a_\alpha a_\gamma \pmod{\mathfrak{c}}$. Write $\gamma = pw$ with $p \in \Gamma_\infty$ and $w \in \Gamma_\infty\backslash\Gamma$. Then $\alpha\gamma = \alpha w$ in $\Gamma_\infty\backslash\Gamma$ and the map $\alpha \mapsto \alpha w^{-1}$ is both well defined and a bijection on $\Gamma_\infty\backslash\Gamma$. The proof of **(i)** is then concluded by putting all of this together into the following calculation:

$$\begin{aligned} G_\tau(\gamma z) &= j_\gamma^k(z)\psi_{\mathfrak{c}}(|a_\gamma|) \sum_{\alpha w^{-1} \in \Gamma_\infty\backslash\Gamma} \psi_{\mathfrak{c}}^{-1}(|a_\alpha|) j_\alpha^k(z)^{-1} e(\mathrm{tr}(\tau\alpha z)) \\ &= j_\gamma^k(z)\psi_{\mathfrak{c}}(|a_\gamma|)G_\tau(z). \end{aligned}$$

The proof of **(ii)** follows in the same manner of the integral-weight case, seen in [Stu81, p. 332], and we outline this. By definition

$$\int_Y c_F(\tau,y)e^{-2\pi\,\mathrm{tr}(\tau y)}|y|^{k-1-n}dy = \int_Y\int_X F(z)e(-\mathrm{tr}(\tau\bar{z}))|y|^k d^\times z. \tag{3.2.3}$$

Note $X\times Y$ is a fundamental domain for $\Gamma_\infty\cap\Gamma[1,1]\backslash\mathbb{H}_n$ and integrating $F(z)$ over $\Gamma_\infty\backslash\mathbb{H}_n$ is $N(\mathfrak{b}^{-1})^{\frac{n(n+1)}{2}}$ many integrals of $F(z)$ over $X\times Y$. Choose a set of representatives α for $\Gamma_\infty\backslash\Gamma$ so that $\Gamma_\infty\backslash\mathbb{H}_n = \bigsqcup_\alpha \alpha(\Gamma\backslash\mathbb{H}_n)$. Then the transformation formulae for F and $|y|$ give that the right-hand side of (3.2.3) is equal to

$$N(\mathfrak{b})^{\frac{n(n+1)}{2}}\int_{\Gamma\backslash\mathbb{H}_n} F(z)\sum_{\alpha\in\Gamma_\infty\backslash\Gamma}\overline{\psi_{\mathfrak{c}}^{-1}(|a_\alpha|)j_\alpha^k(z)^{-1}}e(-\mathrm{tr}(\tau\alpha\bar{z}))|y|^k d^\times z,$$

which gives **(ii)**.

For **(iii)** note that $|y|^{\frac{k}{2}}|f(z)|$ is bounded – this can be deduced from the integral-weight case by squaring f (we prove this in Corollary 3.3.2 below). So f is of bounded growth, and using **(ii)** with $F=f$ gives

$$N(\mathfrak{b})^{\frac{n(n+1)}{2}}\,\mathrm{Vol}(\Gamma\backslash\mathbb{H}_n)\langle f, G_\tau\rangle = c_f(\tau,1)\int_Y e^{-2\pi\,\mathrm{tr}(\tau y)}|y|^{k-1-n}dy,$$

from which we obtain **(iii)**. □

The proof of Theorem 3.2.2 now follows precisely as in [Stu81, pp. 332–333], and we outline it again for clarity's sake. Let $\{f_1,\dots,f_d\}$ be an orthonormal basis for $\mathcal{S}_k(\Gamma,\psi)$ and define

$$\widetilde{K}(z,w) := \sum_{i=1}^d f_i(z)\overline{f_i(w)}.$$

This is precisely the function satisfying

$$\widetilde{K}(z,w)\in\mathcal{S}_k(\Gamma,\psi) \text{ for each } w\in\mathbb{H}_n, \tag{3.2.4}$$

$$\langle g(z),\widetilde{K}(z,w)\rangle = g(w) \text{ for any } g\in\mathcal{S}_k(\Gamma,\psi). \tag{3.2.5}$$

Now define

$$K(z,w) := N(\mathfrak{b})^{\frac{n(n+1)}{2}}\,\mathrm{Vol}(\Gamma\backslash\mathbb{H}_n)\mu(k,n)^{-1}\sum_{0<\tau\in S_{\mathfrak{b}}^\triangledown}|4\tau|^{k-\frac{n+1}{2}}G_\tau(z)e(-\mathrm{tr}(\tau\bar{w})).$$

By Proposition 3.2.3 **(i)** and **(ii)** the function $K(z,w)$ satisfies (3.2.4) and (3.2.5), and therefore $K(z,w)=\widetilde{K}(z,w)$. The projection map in question is defined by

$$\mathbf{Pr}(F)(w) = \langle F(z),K(z,w)\rangle = \sum_{i=1}^d\langle F(z),f_i(z)\rangle f_i(w)\in\mathcal{S}_k(\Gamma,\psi),$$

and clearly we have $\langle\mathbf{Pr}(F),g\rangle = \langle F,g\rangle$ for any g as in Theorem 3.2.2. The explicit form for the Fourier coefficients given in Theorem 3.2.2 can be seen by using the definition of $K(z,w)$ in $\langle F(z),K(z,w)\rangle$ and by using Proposition 3.2.3 **(ii)**.

The moderate-growth case is a bit more complicated, but the core of the above idea remains unchanged. As we saw above, the only real difference in this case was setting up the Poincaré series properly so that Proposition 3.2.3 **(ii)** is satisfied – the rest then follows as in the integral-weight case. In this spirit then, we define the half-integral weight double-variable Poincaré and leave most of the details to the four-page proof seen in [Pan91, pp. 72–76]. For variables $z, w \in \mathbb{H}_n$ and $s \in \mathbb{C}$ define

$$P(z,w,s) := (\Delta(z)\Delta(w))^s \sum_{\gamma\in\Gamma} \psi_{\mathfrak{c}}^{-1}(|a_\gamma|) j_\gamma^k(z)^{-1} \|\mu(\gamma,z)\|^{-2s} |\gamma\cdot z + w|^{-k} \|\gamma \cdot z + w\|^{-2s}.$$

This converges absolutely and uniformly on products $V(d)\times V(d)$ where $\mathfrak{Re}(2s) > 2m-k+1$, $d > 0$, and $V(d) = \{z \in \mathbb{H}_n \mid y \geq dI_n, \mathrm{tr}(x^T x) \leq d^{-1}\}$, see [Pan91, p. 72]

Proposition 3.2.4. *For any $\gamma \in \Gamma$ let $\gamma' := \left(\begin{smallmatrix} I_n & 0 \\ 0 & -I_n \end{smallmatrix}\right) \gamma^{-1} \left(\begin{smallmatrix} I_n & 0 \\ 0 & -I_n \end{smallmatrix}\right)$. We have*

$$j_\gamma^k(z)|\gamma z + w|^k = j_{\gamma'}^k(w)|\gamma' w + z|^k.$$

Proof. By routine calculation $||\mu(\gamma,z)(\gamma z + w)|^\kappa| = ||\mu(\gamma',w)(\gamma' w + z)|^\kappa|$ for any $\kappa \in \frac{1}{2}\mathbb{Z}$. We claim that

$$\frac{h_{\gamma'}(w)}{|h_{\gamma'}(w)|} = \frac{h_\gamma(z)}{|h_\gamma(z)|} \in \mathbb{T}.$$

These are constants independent of w and z respectively. Proposition 2.6 of [Shi93] gives that

$$\frac{h((\gamma')^{-1},w)}{|h((\gamma')^{-1},w)|} = \overline{\frac{h_\gamma(z)}{|h_\gamma(z)|}},$$

since $\gamma' = (\gamma^*)^{-1}$ in Shimura's notation. Using the cocycle relation of (2.1.3) we obtain $h_{\gamma'}(z) = h((\gamma')^{-1}, z)^{-1}$, which gives the claim and so

$$\begin{aligned} h_\gamma(z)|\gamma z + w|^{\frac{1}{2}} &= \frac{h_\gamma(z)}{|h_\gamma(z)|} ||\mu(\gamma,z)(\gamma z + w)|^{\frac{1}{2}}| \\ &= \frac{h_{\gamma'}(w)}{|h_{\gamma'}(w)|} ||\mu(\gamma',w)(\gamma' w + z)|^{\frac{1}{2}}| \\ &= h_{\gamma'}(z)|\gamma' w + z|^{\frac{1}{2}}, \end{aligned}$$

which gives the proposition. □

We have the following properties

$$P(z,w,s) = P(w,z,s), \tag{3.2.6}$$

$$P(\gamma_1 z, \gamma_2 w, s) = \psi_{\mathfrak{c}}(|a_{\gamma_1} a_{\gamma_2}|) j_{\gamma_1}^k(z) j_{\gamma_2}^k(w) P(z,w,s), \tag{3.2.7}$$

$$\langle F(w), P(-\bar{z}, w, s)\rangle = \mu F(z), \tag{3.2.8}$$

for any $F \in C_k^\infty(\Gamma,\psi)$ such that the integral of (3.2.8) converges and for some constant μ given in [Pan91, p. 73]. Property (3.2.6) is immediate from Proposition 3.2.4. The transformation of (3.2.7) is obvious in the z-variable, and the w-variable is due to the first property (3.2.6). The integral expression of (3.2.8) perhaps needs a little explaining. Let

$u^{-k-|2s|} := u^{-k}|u|^{-2s}$ for any $u \in \mathbb{C}$, then the integral $\langle F(z), P(z, -\bar{w}, s)\rangle$ of (3.2.8) is

$$(-1)^{ns}\Delta(w)^s \int_{\Gamma\backslash\mathbb{H}_n} \sum_{\gamma\in\Gamma} \psi_{\mathfrak{c}}(|a_\gamma|)F(z)\overline{j_\gamma^k(z)^{-1}}\|\mu(\gamma,z)\|^{-2s}|\gamma\bar{z} - w|^{-k-|2s|}\Delta(z)^{k+s}d^\times z.$$

Now use that $\psi_{\mathfrak{c}}(|a_\gamma|)F(z) = j_\gamma^k(z)^{-1}F(\gamma\cdot z)$ and $\Delta(z) = \|\mu(\gamma,z)\|^2\Delta(\gamma z)$ to get

$$\begin{aligned}(-1)^{ns}\Delta(w)^s &\int_{\Gamma\backslash\mathbb{H}_n} \sum_{\gamma\in\Gamma} F(\gamma w)|\gamma\bar{w} - z|^{-k-|2s|}\Delta(\gamma z)^{k+s}d^\times z \\ &= (-1)^{ns}\Delta(w)^s \int_{\mathbb{H}_n} F(w)|\bar{w} - z|^{-k-|2s|}\Delta(z)^{k+s}d^\times z\end{aligned}$$

which is exactly of the form found in (4.14) of [Pan91, p. 73]. Therefore the rest of that proof now applies and we obtain (3.2.8). Note that the above integral is convergent and has analytic continuation to $s = 0$ precisely when F is of moderate growth.

To finish the projection in this case set $K(z,w,s) = \mu^{-1}P(-\bar{z}, w, s)$ and one plays a similar game to the bounded-growth case by setting $\mathbf{Pr}(F)(z) := \langle F(w), K(z,w,s)\rangle|_{s=0}$, which we are able to do by the definition of moderate growth.

3.3 Bounded & moderate growth of Eisenstein series

Notation. k – half-integral weight; $[k] = k - \frac{1}{2} \in \mathbb{Z}$.
$\mathfrak{b}$ – fractional ideal of $\mathbb{Q}$; $\mathfrak{c}$ – integral ideal of $\mathbb{Z}$.
$(\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}) \subseteq 2\mathbb{Z}\times 2\mathbb{Z}$.
$\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$.
Hecke character $\psi : \mathbb{I}_{\mathbb{Q}}/\mathbb{Q}^\times \to \mathbb{T}$ satisfying (2.1.5) and (2.1.6) with $\kappa = k$.
$\delta = n \pmod 2 \in \{0,1\}$.

In this section we extend the bounds obtained by Sturm in [Stu81, pp. 333–336] to the present setting and these bounds shall govern when holomorphic projection is applicable to certain modular forms. For an integral or half-integral weight $\kappa \in \frac{1}{2}\mathbb{Z}$, a congruence subgroup Γ_0 (contained in $\mathfrak{M}$ if $\kappa \notin \mathbb{Z}$), $z \in \mathbb{H}_n$, and $b \in \mathbb{R}$ such that $b > \frac{n+1}{2}$ we define the following majorant of $|E_\kappa(z,b)|$:

$$H_\kappa(z,b;\Gamma_0) = H_\kappa(z,b) := |y|^{b-\frac{\kappa}{2}} \sum_{\alpha\in P\cap\Gamma_0\backslash\Gamma_0} \|c_\alpha z + d_\alpha\|^{-2b}.$$

Now fix a fundamental domain Φ of $Sp_n(\mathbb{Z})\backslash\mathbb{H}_n$, chosen so that $z = x + iy \in \Phi$ implies $y > \varepsilon I_n$ for some $\varepsilon > 0$ independent of z.

Proposition 3.3.1. *Let $C_0, a \in \mathbb{R}$ be given, with $C_0 > 0$ and $a \geq 0$. Let $\varphi : \mathbb{H}_n \to \mathbb{C}$ be such that*

$$|\varphi^2(\gamma\cdot z)| \leq C_0|y|^a,$$

for all $z \in \Omega$ and $\gamma \in Sp_n(\mathbb{Z})$. Then, writing λ_j as the eigenvalues of y and taking only *positive square roots, we have*

$$|\varphi(z)| \leq C_1 \prod_{j=1}^{n} (\lambda_j^{\frac{a}{2}} + \lambda_j^{-\frac{a}{2}}),$$

for some constant $C_1 > 0$ *dependent only on* φ.

Proof. Let $z \in \mathbb{H}_n$ and choose $\gamma \in Sp_n(\mathbb{Z})$ such that $\gamma \cdot z \in \Phi$. Then

$$|\varphi^2(z)| = |\varphi^2(\gamma^{-1}(\gamma \cdot z))| \leq C_0 |\mathfrak{Im}(\gamma \cdot z)|^a = C_0 |y|^a \|c_\gamma z + d_\gamma\|^{-2a}. \tag{3.3.1}$$

Let r be the rank of c_γ; since $c_\gamma d_\gamma^T$ is symmetric there exist $U_1, U_2 \in GL_n(\mathbb{Z})$ such that

$$c_\gamma = U_1 \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} U_2^T, \qquad d_\gamma = U_1 \begin{pmatrix} d_1 & 0 \\ 0 & I_{n-r} \end{pmatrix} U_2^{-1},$$

where $c_1, d_1 \in M_r(\mathbb{Z})$ are such that $|c_1| \neq 0$ and $c_1 d_1^T$ is symmetric. Put $U_2 = \begin{pmatrix} Q & Q' \end{pmatrix}$ where $Q \in M_{n,r}(\mathbb{Z})$ and $Q' \in M_{n,n-r}(\mathbb{Z})$. Then, using $\|U_i\| = 1$ to remove U_1 from the left-hand side of $\|c_\gamma z + d_\gamma\|$ and perform similar elementary manipulations with U_2, we get

$$\|c_\gamma z + d_\gamma\| = \|c_1 Q^T z Q + d_1\| = \|Q^T z Q + c_1^{-1} d_1\| \geq \|Q y Q^T\|,$$

which, with (3.3.1) above, gives

$$|\varphi^2(z)| \leq C_0 |y|^a |y_0|^{-2a},$$

for $y_0 = Q y Q^T$; hence $|\varphi(z)| \leq \sqrt{C_0} |y|^{\frac{a}{2}} |y_0|^{-a}$. In [Stu81, p. 334], Sturm proves that there exist $1 \leq j_1 \leq \cdots \leq j_r \leq n$ such that $|y_0| \geq \alpha \prod_{\nu=1}^{r} \lambda_{j_r}$. As $y > 0$ we have $\lambda_j > 0$ for all j, and so $\prod_{\nu=1}^{r} \lambda_{j_\nu}^{-a} \leq \prod_{j=1}^{n} (1 + \lambda_j^{-a})$ – the left-hand side is just one term in the expansion of the right-hand side, all terms of which are ≥ 0. So, for $C_1 = \sqrt{C_0} \alpha^{-a}$, we get

$$|\varphi(z)| \leq C_1 \prod_{j=1}^{n} \lambda_j^{\frac{a}{2}} (1 + \lambda_j^{-a}) = C_1 \prod_{j=1}^{n} (\lambda_j^{\frac{a}{2}} + \lambda_j^{-\frac{a}{2}}).$$

$\square$

Corollary 3.3.2. *Let* $f \in \mathcal{S}_k(\Gamma, \psi)$; $g \in \mathcal{M}_\ell(\Gamma, \psi)$; $\ell, \kappa \in \frac{1}{2}\mathbb{Z}$; *and* $b > \frac{n+1}{2}$. *Then there exists a constant* $0 < C \in \mathbb{R}$ *such that*

(i)

$$|f(z)| \leq C |y|^{-\frac{k}{2}},$$

(ii)

$$|g(z)| \leq C \prod_{j=1}^{n} (1 - \lambda_j^{-\ell}),$$

(iii)

$$|H_\kappa(z, b)| \leq C \prod_{j=1}^{n} (\lambda_j^{b - \frac{\kappa}{2}} + \lambda_j^{-b - \frac{\kappa}{2}}).$$

Proof. To prove **(i)** consider f^2 – a cusp form of integral weight $2k$ and level Γ. It is known that $|f^2(z)||y|^k$ is bounded for all $z \in \mathbb{H}_n$, hence we can apply Proposition 3.3.1 with $\varphi(z) := |y|^{\frac{k}{2}} f(z)$ and $a = 0$.

Likewise, in **(ii)**, Consider g^2 – a modular form of integral weight 2ℓ and level Γ. Then $\varphi(z) := |y|^{\frac{\ell}{2}} g(z)$ satisfies the conditions of Proposition 3.3.1 with $a = \frac{2\ell}{2} = \ell$.

(iii) Sturm shows in [Stu81, p. 335] and Equation (12) in particular that if $\kappa \in \mathbb{Z}$ and $\varphi(z) := |y|^{\frac{\kappa}{2}} H_\kappa(z, b)$, then $|\varphi(\gamma \cdot z)| \leq C_0 |y|^b$ for any $\gamma \in Sp_n(\mathbb{Z})$ and $z \in \Phi$. Now taking $\kappa \in \frac{1}{2}\mathbb{Z}$ we have that $H_\kappa^2(z, b)$ is a constant multiple of $H_{2\kappa}(z, 2b)$, which is of integral weight. Hence, as defined, φ satisfies the conditions of Proposition 3.3.1 with $a = 2b$. $\square$

Corollary 3.3.3. *Let k be a half-integral weight, $\ell \in \frac{1}{2}\mathbb{Z}$, $g \in \mathcal{M}_\ell(\Gamma, \psi)$, $b > \frac{n+1}{2}$, and put $F^*(z) := g(z) H_{k-\ell}(z, b)$. Then F^* is of* bounded growth *provided*

$$\frac{n+1}{2} < b < \begin{cases} \frac{k}{2} - n & \text{if } g \in \mathcal{S}_\ell(\Gamma, \psi), \\ \frac{k-\ell}{2} - n & \text{otherwise.} \end{cases}$$

The proof of the above corollary is exactly the same as in [Stu81, pp. 335–336] and involves using the bounds of Corollary 3.3.2 to determine precisely when the integral of (3.2.1) is finite. We do not reproduce it here since we shall next give the proof of the moderate-growth case, which is similar.

Corollary 3.3.4. *Let k be a half-integral weight, $\ell \in \frac{1}{2}\mathbb{Z}$, $g \in \mathcal{M}_\ell(\Gamma, \psi)$, $b > \frac{n+1}{2}$, and put $F^*(z) := g(z) H_{k-\ell}(z, b)$. Then F^* is of* moderate growth *provided*

$$\tfrac{k+\ell}{2} - n(k+1) - 2 < b < n(k+1) - \tfrac{k+\ell}{2} + 2.$$

Proof. Set $s = 0$ in the integral defining moderate growth, see (3.2.2) in Definition 3.2.1. In terms of the bounds that will be obtained this is perhaps an excessive way to guarantee analytic continuation to $s = 0$, but it is enough for our purposes. Fix $z \in \mathbb{H}_n$, let $w = x + iy$, and let λ_j be the eigenvalues of y. Notice that $|\bar{w} - z|$ is a polynomial in x_{ij}, y_{ij} of degree $n > 0$, which is $|-iy - z|$ as $x \to 0$. Hence $\|\bar{w} - z\|^{-k}$ decays as $|x| \to \pm\infty$, and is finite when $x \to 0$. So then, by Corollary 3.3.2, for some constant ν we may write

$$\int_{\mathbb{H}_n} |F^*(w)| \|\bar{w} - z\|^{-k} |y|^{k-n-1} dy dx \leq \nu \int_Y \upsilon(y) |P(y)|^{-k} dy,$$

where $P(y)$ is a polynomial in y_{ij} of degree n, and

$$\upsilon(y) := \prod_{k=1}^{n} (1 - \lambda_j^{-\ell}) \left(\lambda_j^{b - \frac{k-\ell}{2}} + \lambda_j^{-b - \frac{k-\ell}{2}} \right) \lambda_j^{k-1-n}.$$

Let $\widetilde{\Lambda} := \{\mathrm{diag}[\lambda_1, \cdots, \lambda_n] \mid 0 < \lambda_1 \leq \cdots \leq \lambda_n\}$. As is done in the proof of Corollary 2 of [Stu81, p. 336], we can make the substitution $y = U \Lambda U^T$, where $U \in O_n(\mathbb{R})$ is orthogonal and $\Lambda \in \widetilde{\Lambda}$. If λ_i are all distinct then this is unique up to multiplication of U by $\mathrm{diag}[\pm 1, \ldots, \pm 1]$. Now $\upsilon(y)$ and the Jacobian of the transformation are independent of

U. Therefore the integral over $O_n(\mathbb{R})$ will be finite so long as the integral over $\widetilde{\Lambda}$ is, for which it is enough to show that

$$\int_{\widetilde{\Lambda}} \upsilon(y)|P(\Lambda)|^{-k}|J(\lambda_1,\ldots,\lambda_n)|d\lambda_1\cdots d\lambda_n < \infty,$$

where $J(\lambda_1,\ldots,\lambda_n)$ is the Jacobian matrix of the transformation. To do this, we need only check the limits $\lambda_j \to 0$ and $\lambda_j \to \infty$. Firstly, as $\lambda_j \to 0$ we have $|P(\Lambda)|^{-k} \to \|z\|^{-k}$ is just finite, so the exponent of each λ_j in the integrand needs to be > -1 (and recall $b > \frac{n+1}{2}$). This just gives the original bounds found in Corollary for bounded growth. For the limit $\lambda_j \to \infty$ we have that $|P(y)|^{-k}$ decays to order nk in λ_j, so as long as the exponent of λ_j in $\upsilon(y)$ is $\leq nk$ we obtain convergence. That is

$$\begin{aligned} b - \tfrac{k-\ell}{2} + k - 1 - n &\leq nk, \\ -b - \tfrac{k-\ell}{2} + k - 1 - n &\leq nk, \end{aligned}$$

which gives the desired bound

$$\tfrac{k+\ell}{2} - n(k+1) - 2 < b < n(k+1) - \tfrac{k+\ell}{2} + 2.$$

□

3.4 Projecting the Rankin-Selberg integral

Notation. k – half-integral weight; $[k] = k - \frac{1}{2} \in \mathbb{Z}$.
$\mathfrak{b}$ – fractional ideal of $\mathbb{Q}$; $\mathfrak{c}$ – integral ideal of $\mathbb{Z}$.
$(\mathfrak{b}^{-1}, \mathfrak{bc}) \subseteq 2\mathbb{Z} \times 2\mathbb{Z}$.
$\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}]$.
Hecke character $\psi : \mathbb{I}_{\mathbb{Q}}/\mathbb{Q}^\times \to \mathbb{T}$ satisfying (2.1.5) and (2.1.6) with $\kappa = k$.
$\delta = n \pmod 2 \in \{0,1\}$.
$S_+ = \{\xi \in M_n(\mathbb{Q}) \mid \xi^T = \xi, \xi \geq 0\}$; $\tau \in S_+$.
$S^\triangledown$ – set of symmetric half-integral $n \times n$ matrices.
$S^\triangledown_{\mathfrak{b}}$ – set of symmetric $n \times n$ matrices such that $\tau \in N(\mathfrak{b})S^\triangledown$.
ρ_τ – quadratic character associated to $\mathbb{Q}(i^{[n/2]}\sqrt{|2\tau|})/\mathbb{Q}$.

The results in the previous section tell us when we are able to project the Eisenstein series appearing in the Rankin-Selberg integral expression of (2.4.4). Using Theorem 3.1.1 and the precise description of the Fourier coefficients under projection we are able to determine algebraicity of the resultant holomorphic form and give a useful modification of the integral expression of (2.4.4), which we do in this section. The case of bounded growth of the Eisenstein series is done first, followed by the moderate-growth case.

Lemma 3.4.1. *Let $0 < \tau \in S^\triangledown_{\mathfrak{b}}$ and let $P(y) \in \mathbb{Q}[y_{ij} \mid i \leq j]$. If $\varsigma \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$ is such that $\varsigma > n$ then*

$$|\tau|^{\frac{n}{2}}\Gamma_n\left(\varsigma - \tfrac{n+1}{2}\right)^{-1}\int_Y P(y)e^{-\operatorname{tr}(\tau y)}|y|^{\varsigma - \frac{n+1}{2}}d^\times y \in \mathbb{Q}.$$

Proof. We can assume $P(y) = \prod_{i\le j} y_{ij}^{a_{ij}}$ is a monomial, where $0 \le a_{ij} \in \mathbb{Z}$. If $\upsilon = (\upsilon_{ij}) \in Y$ then by definition

$$\int_Y |y|^{\varsigma - \frac{n+1}{2}} e^{-\operatorname{tr}(\upsilon y)} d^\times y = \Gamma_n\left(\varsigma - \tfrac{n+1}{2}\right) |\upsilon|^{\frac{n+1}{2} - \varsigma},$$

and apply $\prod_{i\le j}\left(\frac{d}{d\upsilon_{ij}}\right)^{a_{ij}}\big|_{\upsilon_{ij}=\tau_{ij}}$ to both sides. This gives

$$\int_Y P(y) e^{-\operatorname{tr}(\tau y)} |y|^{\varsigma - \frac{n+1}{2}} d^\times y \in \Gamma_n\left(\varsigma - \tfrac{n+1}{2}\right) |\tau|^{\frac{n+1}{2} - \varsigma}\mathbb{Q},$$

which, since $\frac{1}{2} - \varsigma \in \mathbb{Z}$, gives the lemma. □

Let $\ell \in \frac{1}{2}\mathbb{Z}$ and let $g \in \mathcal{M}_\ell(\Gamma', \psi')$, where $\Gamma' \le \mathfrak{M}$ if $\ell \notin \mathbb{Z}$ and ψ' is a normalised Hecke character satisfying the usual properties, (2.1.5) and (2.1.6) with $\kappa = \ell$, and further assume

$$(\psi/\psi')_\infty(x) = \operatorname{sgn}(x_\infty)^{[k-\ell]};$$

this latter assumption is only strictly necessary when n is even, since the property (2.1.6) of both ψ and ψ' guarantees it when n is odd, and it ensures that ψ/ψ' satisfies (2.1.7) and we can therefore define the Eisenstein series

$$\mathcal{E}_{k-\ell}^{\bar\psi\psi'}(z, s) := \mathcal{E}_{k-\ell}(z, s; \bar\psi\psi', \Gamma \cap \Gamma').$$

Now define the sets

$$\Omega'_g := \begin{cases} \left\{ m \in \frac{1}{2}\mathbb{Z} \middle| \frac{n-2m+2k-2\ell}{4} \in \mathbb{Z}, \frac{3n}{2} + 1 < m < k - \ell - \frac{3n}{2} \right\} & \text{if } g \notin \mathcal{S}_\ell, \\ \left\{ m \in \frac{1}{2}\mathbb{Z} \middle| \frac{n-2m+2k-2\ell}{4} \in \mathbb{Z}, \frac{3n}{2} + 1 < m < k - \frac{3n}{2}, m \le k - \ell + \frac{n}{2} \right\} & \text{if } g \in \mathcal{S}_\ell. \end{cases}$$

Recall the definitions of β_m, r, and the period ω_ℓ from Theorem 3.1.1.

Proposition 3.4.2. *Let $k > 2n$, let ℓ and g be as above, and take $m \in \Omega'_g$. Set*

$$m_0 := \tfrac{2k+2\ell+2m-n}{4} - \tfrac{n+1}{2}.$$

Then there exists $K(m, g) \in \mathcal{S}_k(\Gamma, \psi)$, whose Fourier coefficients lie in $\mathbb{Q}_{ab}(g)$, such that

$$\frac{(4\pi)^{nm_0}}{\pi^{\beta_m}\omega_\ell(m, \bar\psi\psi')}\Gamma_n(m_0)^{-1}\left\langle f, g\mathcal{E}_{k-\ell}^{\bar\psi\psi'}(\cdot, \tfrac{2m-n}{4})\right\rangle = \pi^{n(k-r) - \frac{3n^2+2n+\delta}{4}}\langle f, K(m, g)\rangle$$

for all $f \in \mathcal{S}_k(\Gamma, \psi)$. We have $K(m, g)^\sigma = K(m, g^\sigma)$ for any $\sigma \in \operatorname{Aut}(\mathbb{C})$.

Proof. The function $\mathcal{F}(z) := g(z)\mathcal{E}_{k-\ell}^{\bar\psi\psi'}(z, \frac{2m-n}{4}) \in C_k^\infty(\Gamma, \psi)$ has bounded growth precisely when $F(z) := g(z)E_{k-\ell}(z, \frac{2m-n}{4}; \bar\psi\psi', \Gamma \cap \Gamma')$ does, and since

$$|E_{k-\ell}(z, \tfrac{2m-n}{4}; \bar\psi\psi', \Gamma \cap \Gamma')| \le H_{k-\ell}(z, \tfrac{2m-n}{4}; \Gamma \cap \Gamma')$$

we invoke Corollary 3.3 to determine this. That $\frac{2m-n}{4}$ satisfies the bounds of said corollary is clear by definition of Ω'_g. The additional bound for the case $g \in \mathcal{S}_\ell$ is there to ensure that $\frac{2m-n}{4}$ belongs to the set Ω_0 defined in (3.1.1), which is needed to apply Theorem 3.1.1.

In accordance with Theorem 3.2.2, set $\widetilde{K}(m,g) := \frac{(4\pi)^{nm_0}}{\pi^{\beta}\omega(\bar{\psi}\psi')}\Gamma_n(m_0)^{-1}\mathbf{Pr}(\mathcal{F}) \in \mathcal{S}_k(\Gamma,\psi)$ and we have

$$\frac{(4\pi)^{nm_0}}{\pi^{\beta_m}\omega_\ell(m,\bar{\psi}\psi')}\Gamma_n(m_0)^{-1}\left\langle f, g\mathcal{E}_{k-\ell}^{\bar{\psi}\psi'}(\cdot,\tfrac{2m-n}{4})\right\rangle = \langle f, \widetilde{K}(m,g)\rangle.$$

We now use the explicit expression of the Fourier coefficients in Theorem 3.2.2 to show that $\widetilde{K}(m,g)$ has coefficients in $\pi^{nk-\frac{3n^2+2n+\delta}{4}}\mathbb{Q}_{ab}(g)$, from which we obtain the proof. By Theorem 3.1.1 the Fourier coefficients of $\mathcal{F}$ are given as

$$c_{\mathcal{F}}(\tau,y) = \sum_{\tau_1+\tau_2=\tau} c_g(\tau_1,1)P(\tau_2,\bar{\psi}\psi',y)|\pi y|^{-r}e^{-2\pi\operatorname{tr}(\tau y)},$$

where in this case $r = \frac{n-2m+2k-2\ell}{4}$. So, by Theorem 3.2.2, we have

$$\begin{aligned}
\widetilde{K}(m,g) &= \sum_{0<\tau\in S_{\mathfrak{b}}^{\triangledown}}\left(\sum_{\tau_1+\tau_2=\tau} a(\tau_1,\tau_2)\right)e(\operatorname{tr}(\tau z)),\\
a(\tau_1,\tau_2) &= \mu(k,n)^{-1}c_g(\tau_1,1)|4\tau|^{k-\frac{n+1}{2}}\frac{(4\pi)^{nm_0}}{\pi^{\beta+nr}\omega_\ell(\bar{\psi}\psi')}\Gamma_n(m_0)^{-1}\\
&\quad\times\int_Y P(\tau_2,\bar{\psi}\psi',y)e^{-4\pi\operatorname{tr}(\tau y)}|y|^{k-r-\frac{n+1}{2}}d^\times y.
\end{aligned}$$

The polynomial $P(\tau_2,\bar{\psi}\psi',y)$ has the form

$$P(\tau_2,\bar{\psi}\psi',y) = \sum_{1\le i\le j\le n}\sum_{\alpha_{ij}} b_{ij}(\tau_2,\bar{\psi}\psi')\prod_{i,j}(\pi y_{ij})^{\alpha_{ij}},$$

for coefficients $b_{ij} = b_{ij}(\tau_2,\bar{\psi}\psi') \in \mathbb{Q}_{ab}$ and α_{ij} summing over some finite set of non-negative integers. We have $|4\tau|^{k-\frac{n+1}{2}} \in |\tau|^{\frac{n}{2}}\mathbb{Q}$ and by routine calculation we see that $\mu(k,n)^{-1} \in \pi^{nk-\frac{3n^2+2n+\delta}{4}}\mathbb{Q}$. Note also $m_0 = k-r-\frac{n+1}{2}$ and $k-r>n$; the substitution $y \mapsto (4\pi)^{-1}y$ then gives

$$\begin{aligned}
a(\tau_1,\tau_2) &\in \frac{c_g(\tau_1,1)\pi^{nk-\frac{3n^2+2n+\delta}{4}}}{\pi^{\beta+nr}\omega_\ell(\bar{\psi}\psi')}\sum_{i,j,\alpha_{ij}} b_{ij}\left[|\tau|^{\frac{n}{2}}\Gamma_n(m_0)^{-1}\int_Y\left(\prod_{i,j}y_{ij}^{\alpha_{ij}}\right)e^{-\operatorname{tr}(\tau y)}|y|^{m_0}\right]\mathbb{Q}\\
&\in c_g(\tau_1,1)\pi^{n(k-r)-\frac{3n^2+2n+\delta}{4}}\sum_{i,j}\frac{b_{ij}(\tau_2,\bar{\psi}\psi')}{\pi^{\beta}\omega_\ell(\bar{\psi}\psi')}\mathbb{Q},
\end{aligned}$$

where we used Lemma 3.4.1. The direct application of Theorem 3.1.1 then gives the proposition with $K(m,g) := \pi^{\frac{3n^2+2n+\delta}{4}-n(k-r)}\widetilde{K}(m,g)$. □

The above proposition can be used to modify the original Rankin-Selberg expression of (2.4.4). As in that expression, take a Hecke character $\chi : \mathbb{I}_{\mathbb{Q}}/\mathbb{Q}^\times \to \mathbb{T}$ of conductor $\mathfrak{f}$ and a $\tau \in S_+$ such that $c_f(\tau,1) \neq 0$. Recall $\mathfrak{t}$ as an integral ideal such that $h^T(2\tau)^{-1}h \in 4\mathfrak{t}^{-1}$ for all $h \in \mathbb{Z}^n$ and $\mathfrak{y} = \mathfrak{c}\cap(4\mathfrak{t}\mathfrak{f}^2)$. As usual, take $\mu \in \{0,1\}$ such that $(\psi\chi)_\infty(x) = \operatorname{sgn}(x_\infty)^{[k]+\mu}$.

Now assume that $k>2n$, take $g = \theta_\chi$ and $\ell = \frac{n}{2}+\mu$ in the above proposition, and $m \in \Omega'_{\theta_\chi}$. Firstly, $\left(\frac{\Lambda_{\mathfrak{c}}}{\Lambda_{\mathfrak{y}}}\right)(\frac{2s-n}{4})$ is a finite product, over all $p \mid \mathfrak{y}$ such that $p \nmid \mathfrak{c}$, of Euler factors twisted by $\eta = \psi\chi\rho_\tau$. Hence, denoting it by $\epsilon_{\mathfrak{y}}(s,\eta) = \epsilon_{k,\mathfrak{y}}^n(s,\eta)$, we have $\epsilon_{\mathfrak{y}}(s,\eta)^\sigma = \epsilon_{\mathfrak{y}}(s,\eta^\sigma)$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. We have that $m_0 = \frac{m-n-1+k+\mu}{2} \in \frac{n}{2} + \mathbb{Z}$ occurs naturally in the integral expression of (2.4.4), and moreover $|\tau|^{\frac{1}{2}} \in 2^{\frac{1}{2}}|2\tau|^{-\frac{1}{2}}\mathbb{Q}$ and $\mathrm{Vol}(\Gamma\backslash\mathbb{H}_n) \in \pi^{\frac{n(n+1)}{2}}\mathbb{Q}$ (see [Shi00, p. 231]). Thus the integral expression in (2.4.4) can be rewritten as

$$L_\psi(m,f,\chi) \in (2|2\tau|)^{-\frac{\delta}{2}} c_f(\tau,1)^{-1}\epsilon_{\mathfrak{y}}(m,\eta)\prod_{p\in\mathbf{b}} g_p\left((\psi^{\mathfrak{c}}\chi^*)(p)p^{-m}\right)\pi^{\frac{n(n+1)}{2}}$$
$$\times (4\pi)^{nm_0}\Gamma_n(m_0)^{-1}\left\langle f, \theta_\chi \mathcal{E}^{\bar{\eta}}_{k-\frac{n}{2}-\mu}(\cdot, \tfrac{2m-n}{4})\right\rangle \mathbb{Q}.$$

Multiplying both sides by $\pi^{-\beta_m}\omega_\delta(m,\bar{\eta})^{-1}$ and applying Proposition 3.4.2 we get

$$\frac{(2|2\tau|)^{\frac{\delta}{2}}c_f(\tau,1)L_\psi(m,f,\chi)}{\pi^{\beta_m+n(k-r)-\frac{n^2+\delta}{4}}\omega_\delta(m,\bar{\eta})} \in \epsilon_{\mathfrak{y}}(m,\eta)\prod_{p\in\mathbf{b}} g_p\left((\psi^{\mathfrak{c}}\chi^*)(p)p^{-m}\right)\langle f, K(m,\theta_\chi)\rangle\mathbb{Q}. \qquad (3.4.1)$$

The result of Proposition 3.4.2 is now repeated for the moderate-growth case. In this case define, for any $g \in \mathcal{M}_\ell(\Gamma,\psi')$ as above, the sets

$$\begin{aligned}\Omega_g^+ &:= \{m \in \tfrac{1}{2}\mathbb{Z} \mid \tfrac{n-2m+2k-2\ell}{4} \in \mathbb{Z}, n < m \le k-\ell+\tfrac{n}{2}\},\\ \Omega_g^- &:= \{m \in \tfrac{1}{2}\mathbb{Z} \mid \tfrac{2m-3n+2k-2\ell-2}{4} \in \mathbb{Z}, \tfrac{3n}{2}+1-k+\ell \le m \le n\},\end{aligned} \qquad (3.4.2)$$

and put $\Omega_g := \Omega_g^- \cup \Omega_g^+$

Proposition 3.4.3. *Exclude case (**X**). Let $k > 2n$, ℓ, and g be as in Proposition 3.4.2. In case (**R1**) set $m_0 := \frac{k+\ell-3}{2}$ and in case (**R2**) set $m_0 := \frac{2k+2\ell+n-1}{4} - \frac{n+1}{2}$; in all other cases*

$$m_0 := \begin{cases} \frac{2k+2\ell+2m-n}{4} - \frac{n+1}{2} & \text{if } m > n,\\ \frac{2k+2\ell+3n-2m+2}{4} - \frac{n+1}{2} & \text{if } m \le n.\end{cases}$$

For every $m \in \Omega_g$ there exists $K_{\mathcal{S}}(m,g) \in \mathcal{S}_k(\Gamma,\psi)$, whose Fourier coefficients lie in $\mathbb{Q}_{ab}(g,\Lambda_{k,\psi},G(\psi),\zeta_\star)$, such that

$$\frac{(4\pi)^{nm_0}}{\pi^{\beta_m}\omega_\ell(m,\bar{\psi}\psi')}\Gamma_n(m_0)^{-1}\left\langle f, g\mathcal{E}^{\bar{\psi}\psi'}_{k-\ell}(\cdot,\tfrac{2m-n}{4})\right\rangle = \pi^{n(k-r)-\frac{3n^2+2n+\delta}{4}}\langle f, K_{\mathcal{S}}(m,g)\rangle$$

for all $f \in \mathcal{S}_k(\Gamma,\psi)$.

For any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Lambda_{k,\psi},G(\psi),\zeta_\star))$ we have $K_{\mathcal{S}}(m,g)^\sigma = K_{\mathcal{S}}(m,g^\sigma)$.

Proof. The proof of this is pretty much identical to that of Proposition 3.4.2, so we just make some remarks on the different inequalities involved and how this guarantees the proof still works. To apply holomorphic projection we ensure moderate growth of $g(z)H_{k-\ell}(z,\frac{2m-n}{4})$ when $m \in \Omega_g^+$. The analytic continuation of $\mathcal{E}_{k-\ell}(z,s)$ is given with respect to $s \mapsto \frac{n+1}{2} - s$ and so we consider the majorant $H_{k-\ell}(z,\frac{n+1}{2}-\frac{2m-n}{4})$ whenever $m \in \Omega_g^-$. That $\frac{2m-n}{4}$ and $\frac{n+1}{2}-\frac{2m-n}{4}$ satisfy the bounds of Proposition 3.3.4 is immediate from the definition of Ω_g. Moreover $\frac{2m-n}{4} \in \Omega_0$ – allowing the application of Theorem 3.1.1 – and $k-r > n$ – allowing the application of Lemma 3.4.1. The changes in the definition of m_0 in the four separate cases is a result of the change to the order r of the non-holomorphic Eisenstein series – see Theorem 3.1.1.

Therefore in applying holomorphic projection, and replicating the proof of Proposition 3.4.2, we obtain a holomorphic modular form $K(m,g) \in \mathcal{M}_k(\Gamma,\psi)$ with coefficients in $\mathbb{Q}_{ab}(g)$. By Theorem A this splits up as $K(m,g) = K_{\mathcal{S}}(m,g) + K_{\mathcal{E}}(m,g)$ where $K_{\mathcal{X}}(m,g) \in \mathcal{X}_k(\Gamma,\psi)$, for $\mathcal{X} \in \{\mathcal{S},\mathcal{E}\}$, has coefficients in $\mathbb{Q}_{ab}(g, \Lambda_{k,\psi}, G(\psi), \zeta_\star)$. Since $\langle f, K_{\mathcal{E}}(m,g)\rangle = 0$ we are done. □

Now assume that $k > 2n$, take $g = \theta_\chi$ and $\ell = \frac{n}{2} + \mu$ in the above proposition. If $m \in \Omega^+_{\theta_\chi}$ and we are not in cases (**R1**) or (**R2**) then we obtain the same integral expression of (3.4.1) above for $L_\psi(m,f,\chi)$. On the other hand if we are in case (**R1**) or (**R2**), or if $m \in \Omega^-_{\theta_\chi}$, then this will be slightly different since here the value m_0 required to apply Proposition 3.4.3 above is no longer occurring naturally in the original integral expression of (2.4.4). Hence we introduce $(4\pi)^{nm_0}$ and $\Gamma_n(m_0)$ artificially in these cases. If $m \in \Omega^-_{\theta_\chi}$ then

$$m_0 = \tfrac{k+n+\mu-m}{2} = \left(\tfrac{m-n-1+k+\mu}{2}\right) + n - m + \tfrac{1}{2}$$

from which

$$(4\pi)^{n\left(\frac{m-n-1+k+\mu}{2}\right)} = (4\pi)^{nm_0}(4\pi)^{\frac{n}{2}(2m-2n-1)},$$
$$\Gamma_n\left(\tfrac{m-n-1+k+\mu}{2}\right)\Gamma_n(m_0)^{-1} \in \mathbb{Q}.$$

So, from (2.4.4), we obtain

$$L_\psi(m,f,\chi) \in (2|2\tau|)^{-\frac{\delta}{2}}\pi^{\frac{n}{2}(2m-n)}c_f(\tau,1)^{-1}\epsilon_{\mathfrak{y}}(m,\eta)\prod_{p\in\mathbf{b}} g_p\left((\psi^{\mathfrak{c}}\chi^*)(p)p^{-m}\right)$$
$$\times (4\pi)^{nm_0}\Gamma_n(m_0)^{-1}\left\langle f, \theta_\chi\mathcal{E}^{\bar\eta}_{k-\frac{n}{2}-\mu}(\cdot,\tfrac{2m-n}{4})\right\rangle\mathbb{Q}$$

and, as before, multiplying both sides by $\pi^{-\beta_m}\omega_\delta(m,\bar\eta)^{-1}$ and applying Proposition 3.4.3 gives

$$\frac{(2|2\tau|)^{\frac{\delta}{2}}c_f(\tau,1)L_\psi(m,f,\chi)}{\pi^{\beta_m+n(k+m-r)-\frac{5n^2+2n+\delta}{4}}\omega_\delta(m,\bar\eta)} \in \epsilon_{\mathfrak{y}}(m,\eta)\prod_{p\in\mathbf{b}} g_p\left((\psi^{\mathfrak{c}}\chi^*)(p)p^{-m}\right)\langle f, K(m,\theta_\chi)\rangle\mathbb{Q}. \tag{3.4.3}$$

If we are in cases (**R1**) or (**R2**) then $(4\pi)^{n\left(\frac{m-n-1+k+\mu}{2}\right)} = (4\pi)^{nm_0+n}$ and rationality of the Γ-factors is again preserved. Hence for both of these cases we have

$$\frac{(2|2\tau|)^{\frac{\delta}{2}}c_f(\tau,1)L_\psi(m,f,\chi)}{\pi^{\beta_m+n(k-r)-\frac{n^2-4n+\delta}{4}}\omega_\delta(m,\bar\eta)} \in \epsilon_{\mathfrak{y}}(m,\eta)\prod_{p\in\mathbf{b}} g_p\left((\psi^{\mathfrak{c}}\chi^*)(p)p^{-m}\right)\langle f, K(m,\theta_\chi)\rangle\mathbb{Q}. \tag{3.4.4}$$

3.5 Proof of the main theorems

Notation. k – half-integral weight; $[k] = k - \frac{1}{2} \in \mathbb{Z}$.
$\mathfrak{b}$ – fractional ideal of $\mathbb{Q}$; $\mathfrak{c}$ – integral ideal of $\mathbb{Z}$.
$(\mathfrak{b}^{-1}, \mathfrak{bc}) \subseteq 2\mathbb{Z}\times 2\mathbb{Z}$.
$\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}]$.
Hecke character $\psi : \mathbb{I}_{\mathbb{Q}}/\mathbb{Q}^\times \to \mathbb{T}$ satisfying (2.1.5) and (2.1.6).

Notation. $\delta = n \pmod 2 \in \{0,1\}$.
$\varepsilon \in \{0,1\}$ such that $\psi_\infty(x) = \mathrm{sgn}(x_\infty)^{[k]+\varepsilon}$.
Hecke character $\chi : \mathbb{I}_\mathbb{Q}/\mathbb{Q}^\times \to \mathbb{T}$ of conductor $\mathfrak{f}$.
ρ_τ – quadratic character associated to $\mathbb{Q}(i^{[n/2]}\sqrt{|2\tau|})/\mathbb{Q}$.
$\mu \in \{0,1\}$ such that $(\chi\psi)_\infty(x) = \mathrm{sgn}(x_\infty)^{[k]+\mu}$.
$\mathfrak{t}$ – integral ideal such that $h^T(2\tau)^{-1}h \in 4\mathfrak{t}^{-1}$ for all $h \in \mathbb{Z}^n$.
$\mathfrak{y} = \mathfrak{c} \cap (4\mathfrak{t}\mathfrak{f}^2)$.
$\epsilon_\mathfrak{y}(m,\eta) = \left(\frac{\Lambda_\mathfrak{c}}{\Lambda_\mathfrak{y}}\right)\left(\frac{2m-n}{4}\right)$ – finite product of Euler factors.

In the projected integral expressions of the previous section we have managed to express, for certain m, the L-function $L_\psi(s,f,\chi)$ as a Rankin-Selberg integral of two *holomorphic* forms. So it only remains show algebraicity of this Rankin-Selberg integral. As a quotient over a constant, $\mu'(\Lambda,k,\psi)$ of (3.1.2) in the bounded-growth case and $\mu(\Lambda,k,\psi)$ of (3.1.4) in the moderate-growth case, the Petersson inner product becomes entirely equivariant under action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. Recall that $\mathcal{S}_k(\Gamma,\psi,\Lambda)$ denotes the space of eigenforms in $\mathcal{S}_k(\Gamma,\psi)$ with eigenvalues $\Lambda : \mathcal{R}_0^\mathfrak{c} \to \mathbb{C}$. If $f \in \mathcal{S}_k(\Gamma,\psi,\Lambda)$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$ then, by [Shi00, Lemma 23.14], we have $f^\sigma \in \mathcal{S}_k(\Gamma,\psi^\sigma,\Lambda_\sigma)$, where

$$\Lambda_\sigma(A(p^m)) := \Lambda^\sigma(A(p^m))\frac{(\sqrt{p^m})^\sigma}{\sqrt{p^m}},$$

for any prime p. Let $\rho \in \mathrm{Aut}(\mathbb{C})$ denote the complex conjugation automorphism and recall $\varepsilon \in \{0,1\}$ such that $\psi_\infty(x) = \mathrm{sgn}(x_\infty)^{[k]+\varepsilon}$.

Theorem 3.5.1. *Assume that $k > \frac{7n}{2} + 3 + \varepsilon$. If $f \in \mathcal{S}_k(\Gamma,\psi,\Lambda)$ is a Hecke eigenform for a system of eigenvalues $\Lambda : \mathcal{R}_0 \to \mathbb{C}$, then the non-zero constant $\mu'(\Lambda,k,\psi)$ – which is defined by (3.1.2) and is dependent only on $\Lambda, k,$ and ψ – satisfies*

$$\left(\frac{\langle f,g\rangle}{\mu'(\Lambda,k,\psi)}\right)^\sigma = \frac{\langle f^\sigma, g^{\rho\sigma\rho}\rangle}{\mu'(\Lambda_\sigma,k,\psi^\sigma)},$$

for any $g \in \mathcal{S}_k(\Gamma[(\mathfrak{b}')^{-1},\mathfrak{b}'\mathfrak{c}'],\psi)$, ideals $((\mathfrak{b}')^{-1},\mathfrak{b}'\mathfrak{c}') \subseteq \mathfrak{b}^{-1}\times\mathfrak{b}\mathfrak{c}$, and $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$.

Proof. If $\mathfrak{m}$ is an integral ideal let $1_\mathfrak{m} : \mathbb{I}_\mathbb{Q}/\mathbb{Q}^\times \to \mathbb{T}$ denote the Hecke character that is trivial modulo $\mathfrak{m}$, i.e. the associated ideal Hecke character satisfies $1^*_\mathfrak{m}(d) = 1$ for any $(d\mathbb{Z},\mathfrak{m}) = 1$ and $1^*_\mathfrak{m}(d) = 0$ otherwise. Let $\theta_\mathfrak{m}$ denote the theta series of (2.1.9) with $1_\mathfrak{m}$ in place of χ and ε in place of μ. Since $\varepsilon = 0$ unless n is even; the condition $(1_\mathfrak{m})_\infty(x)^n = \mathrm{sgn}(x_\infty)^{n\varepsilon}$ is satisfied, and such a theta series is defined.

Recall $\mathfrak{t}$ as the integral ideal such that $h^T(2\tau)^{-1}h \in 4\mathfrak{t}^{-1}$ for all $h \in \mathbb{Z}^n$ and let $\mathfrak{m} := \mathfrak{t}\mathfrak{c}$, then Proposition 2.1 of [Shi96] gives that $\theta_\mathfrak{m}$ transforms with respect to $\Gamma[2,2\mathfrak{t}^3\mathfrak{c}^2]$. Hence, with $\mathfrak{c}' := 2\mathfrak{b}^{-1}\mathfrak{t}^3\mathfrak{c}^2$, we have $\theta_\mathfrak{m} \in \mathcal{M}_{\frac{n}{2}+\varepsilon}(\Gamma[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}'],\rho_\tau)$ and $f \in \mathcal{S}_k(\Gamma[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}'],\psi)$. Since

$$\Omega'_{\theta_\mathfrak{m}} = \{m \in \tfrac{1}{2}\mathbb{Z} \mid \tfrac{k-m-\varepsilon}{2} \in \mathbb{Z},\ \tfrac{3n}{2}+1 < m < k-2n-\varepsilon\},$$

we have $m_\varepsilon = k-2n-2-\varepsilon \in \Omega'_{\theta_{\varphi_\mathfrak{m}}}$. Now apply Proposition 3.4.2 with $g = \theta_\mathfrak{m}$ to obtain the integral expression, (3.4.1) with $\chi = 1_\mathfrak{m}$, for $L(m_\varepsilon, f, 1_\mathfrak{m})$. The character occurring in the Eisenstein series in this integral expression is $\eta = 1_{\mathfrak{m}}\psi\rho_\tau$, but by definition of $\mathfrak{m}$ we just have $\eta = \psi\rho_\tau$. Recall from Theorem 3.1.1 that the $\omega_\delta(m_\varepsilon, \overline{\psi})^{-1}$, which appears in the definition of $\mu'(\Lambda, k, \psi)$, consists primarily of Gauss sums and we wish to compare $\omega_\delta(m_\varepsilon, \overline{\psi})$ with $\omega_\delta(m_\varepsilon, \overline{\psi}\rho_\tau)$. To this end, notice that

$$G(\overline{\psi}\rho_\tau)G((\overline{\psi}\rho_\tau)^{n-1}) = G(\overline{\psi})G(\overline{\psi})^{n-1}\frac{G(\rho_\tau)^n}{J(\rho_\tau,\rho_\tau)^{n-1}J(\overline{\psi},\rho_\tau)J(\overline{\psi}^{n-1},\rho_\tau^{n-1})},$$

where $J(\chi_1,\chi_2) = \sum_{a \pmod c} \chi_1^*(a)\chi_2^*(1-a)$ is the Jacobi sum of any two Hecke characters χ_1, χ_2 modulo $c\mathbb{Z}$. In the case of n odd and $s > n$ this gives

$$\omega_\delta(\overline{\psi}) = \omega_\delta(\overline{\psi}\rho_\tau)G(\rho_\tau)^{-n}J(\rho_\tau,\rho_\tau)^{n-1}J(\overline{\psi},\rho_\tau)J(\overline{\psi}^{n-1},\rho_\tau^{n-1}).$$

The other three cases of n and s are simpler and give $\omega_\delta(\overline{\psi}) = \omega_\delta(\overline{\psi}\rho_\tau)G(\rho_\tau)^{-n}J(\overline{\psi},\rho_\tau)^n$. In any case, denote these products of Jacobi sums by $\mathcal{J}_n(\psi)$, and evidently we have $\mathcal{J}_n(\psi)^\sigma = \mathcal{J}_n(\psi^\sigma)$. Note also that

$$L_\psi(s,f,1_{\mathfrak{m}}) = R_\psi(s,f)L_\psi(s,f),$$
$$R_\psi(s,f) := \prod_{p|\mathfrak{m}} L_p\left(\psi^{\mathfrak{c}}(p)p^{-s}\right),$$

and put

$$P_\psi(s) := \prod_{p\in\mathbf{b}} g_p\left(\psi^{\mathfrak{c}}(p)p^{-s}\right)^{-1}.$$

The integral expression (3.4.1) for $L_\psi(m_\varepsilon, f, 1_{\mathfrak{m}})$ thus becomes

$$\frac{\langle f, K(m_\varepsilon,\theta_{\mathfrak{m}})\rangle}{\mu'(\Lambda,k,\psi)} \in c_f(\tau,1)\left[\frac{i^{\frac{n}{2}}|2\tau|^{\frac{1}{2}}}{G(\rho_\tau)}\right]^n \frac{\mathcal{J}_n(\psi)P_\psi(m_\varepsilon)}{\epsilon_{\mathfrak{y}}(m_\varepsilon,\psi\rho_\tau)}R_\psi(m_\varepsilon,f)\mathbb{Q},$$

the σ-equivariance of which is easy to see. So for any $\sigma \in \mathrm{Aut}(\mathbb{C})$ we have

$$\left[\frac{\langle f, K(m_\varepsilon,\theta_{\mathfrak{m}})\rangle}{\mu'(\Lambda,k,\psi)}\right]^\sigma = \frac{\langle f^\sigma, K(m_\varepsilon,\theta_{\mathfrak{m}}^\sigma)\rangle}{\mu'(\Lambda_\sigma,k,\psi^\sigma)}.$$

Let

$$\Gamma^0 := \{\gamma \in \Gamma[\mathfrak{x}^{-1},\mathfrak{x}\mathfrak{y}] \mid a_\gamma \equiv d_\gamma \equiv 1 \pmod{\mathfrak{y}}\}.$$

Suppose that $\Gamma_2 \le \Gamma_1 \le \mathfrak{M}$ are two congruence subgroups and decompose $\Gamma_1 = \bigsqcup_{i=1}^d \Gamma_2^0\gamma_i$. The trace map is defined, for any Hecke character ρ, by

$$\begin{aligned}\mathrm{Tr}_{\Gamma_1,\rho}^{\Gamma_2} : \mathcal{M}_k(\Gamma_2^0) &\to \mathcal{M}_k(\Gamma_1,\rho)\\ h &\mapsto \sum_{i=1}^d \rho_{\mathfrak{c}_1}(|a_{\gamma_i}|)^{-1}h||_k\gamma_i,\end{aligned} \tag{3.5.1}$$

where $\Gamma_1 = \Gamma[\mathfrak{b}_1^{-1},\mathfrak{b}_1\mathfrak{c}_1]$. By Lemma 5.4 of [Bou18] we have that $\mathrm{Tr}_{\Gamma_1,\rho}^{\Gamma_2}(h)^\sigma = \mathrm{Tr}_{\Gamma_1,\rho^\sigma}^{\Gamma_2}(h^\sigma)$ for any $h \in \mathcal{M}_k(\Gamma_2^0)$ and any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. Therefore, if $\Gamma' := \Gamma[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}']$ then

$\mathrm{Tr}_{\Gamma,\psi}^{\Gamma'}(K(m_\varepsilon,\theta_{\mathfrak{m}}))\in\mathcal{S}_k(\Gamma',\psi)$. So far we have obtained

$$\left[\frac{\langle f,g\rangle}{\mu'(\Lambda,k,\psi)}\right]^\sigma=\frac{\langle f^\sigma,g^\sigma\rangle}{\mu'(\Lambda_\sigma,k,\psi^\sigma)}$$

for any $g\in\mathfrak{K}:=\left\{\mathrm{Tr}_{\Gamma',\psi}^{\Gamma}(K(m_\varepsilon,\theta_{\mathfrak{m}}))\mid 0<\tau\in S_{\mathfrak{b}}^{\nabla},\mathfrak{m}=\mathfrak{t}\mathfrak{c}\right\}$. This set can be extended into a basis for $\mathcal{S}_k(\Gamma,\psi,\Lambda)$ by the same argument of [Stu81, p. 350]. This basis is defined by

$$\mathfrak{S}:=\{g\in\mathcal{S}_k(\Gamma,\psi,\Lambda)\mid \text{there exists } k_g\in\mathfrak{K} \text{ such that } g-k_g \text{ is orthogonal to } \mathcal{S}_k(\Gamma,\psi,\Lambda)\}.$$

If $g\in\mathcal{S}_k(\Gamma',\psi)\subseteq\mathcal{S}_k(\Gamma,\psi)$ then we can write $g=g_1+g_2$, where $g_1\in\mathcal{S}_k(\Gamma,\psi,\Lambda)$ and $\langle g_1,g_2\rangle=0$; this is so by the decomposition of (2.3.9). So

$$\langle f,g\rangle=\langle f,g_1\rangle=\sum_{s\in\mathfrak{S}}\overline{c_s}\langle f,s\rangle=\sum_{s\in\mathfrak{S}}\overline{c_s}\langle f,k_s\rangle,$$

divide this by $\mu'(\Lambda,k,\psi)$ and apply σ to it. Since each $k_s\in\mathfrak{K}$ we obtain σ-equivariance of $\langle f,k_s\rangle\mu'(\Lambda,k,\psi)^{-1}$ by the argument above. To finish note, for any system of eigenvalues Λ' and any $\sigma'\in\mathrm{Aut}(\mathbb{C})$, that $g^{\sigma'}=g_1^{\sigma'}+g_2^{\sigma'}$ and $\mathcal{S}_k(\Gamma,\psi,\Lambda')^{\sigma'}=\mathcal{S}_k(\Gamma,\psi^{\sigma'},\Lambda'_{\sigma'})$. So by the decomposition (2.3.9) we see $\langle g_1^{\sigma'},g_2^{\sigma'}\rangle$ and this finishes the proof. □

Proof of Theorem B1. The two separate bounds are there to emphasise the fact that $\varepsilon=0$ if n is odd by the property (2.1.6) of ψ (with $\kappa=k$). Noting that $\Omega'_{n,k}=\Omega'_{\theta_\chi}$ and that $(2|2\tau|)^{\frac{\delta}{2}}\in|\tau|^{\frac{\delta}{2}}\mathbb{Q}$, we use the definition of $Y_\psi(m,f,\chi)$ and the integral expression of (3.4.1) to get

$$Y_\psi(m,f,\chi)\in c_f(\tau,1)^{-1}\epsilon_{\mathfrak{y}}(m,\eta)\prod_{p\in\mathbf{b}}g_p\left((\psi^{\mathfrak{c}}\chi^*)(p)p^{-m}\right)\frac{\langle f,K(m,\theta_\chi)\rangle}{\mu'(\Lambda,k,\psi)}\mathbb{Q},$$

which we see is σ-equivariant by Theorem 3.5.1. □

The moderate-growth case is very similar. Theorem B2 improves on Theorem B1 in two ways, the set of special values is expanded and the bounds on the weight k are lowered. Proposition 3.4.3 facilitates the former while the latter arise from the following modification to Theorem 3.5.1.

Theorem 3.5.2. *Assume that $k>\max\{2n,\frac{3n}{2}+1+\varepsilon\}$. If $f\in\mathcal{S}_k(\Gamma,\psi,\Lambda)$ is a Hecke eigenform for a system of eigenvalues $\Lambda:\mathcal{R}_0\to\mathbb{C}$, then the constant $\mu(\Lambda,k,\psi)$ – which defined by (3.1.4) and is dependent only on Λ,k,ψ – satisfies*

$$\left[\frac{\langle f,g\rangle}{\mu(\Lambda,k,\psi)}\right]^\sigma=\frac{\langle f^\sigma,g^{\rho\sigma\rho}\rangle}{\mu(\Lambda_\sigma,k,\psi^\sigma)}$$

for all $\sigma\in\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$.

Proof. Take $\mathfrak{m}$ as in the proof of Theorem 3.5.1; since we have a larger set $\Omega_{\theta_{\mathfrak{m}}}$ of special values defined in (3.4.2), this allows the change to the special value of the L-function in the definition (3.1.4) of $\mu(\Lambda,k,\psi)$. Specifically

$$\Omega^+_{\theta_{\mathfrak{m}}}=\{m\in\tfrac{1}{2}\mathbb{Z}\mid\tfrac{k-m-\varepsilon}{2}\in\mathbb{Z},n<m\le k-\varepsilon\},$$

and since $k - \varepsilon \in \Omega_{\theta_{\mathfrak{m}}}$ the rest of this proof follows exactly as that of Theorem 3.5.1. The bound $k > \frac{3n}{2} + 1 + \varepsilon$ guarantees $\mu(\Lambda, k, \psi) \neq 0$ and the bound $k > 2n$ ensures we can apply holomorphic projection and Proposition 3.4.2; for $n > 2$ we have $2n > \frac{3n}{2} + 1 + \varepsilon$ but for $n \leq 2$ this is no longer necessarily true, so both bounds are needed. $\square$

Remark 3.5.3. The modification above tells us that we could, if we wanted to, simply use the one constant $\mu(\Lambda, k, \psi)$ for both of the main Theorems B1 and B2 and, as a consequence, get the improved bound on k for Theorem B1 as well. They are being kept separate for the following reason: in hindsight we know that – modulo some π-factors, $|\tau|^{\frac{\delta}{2}}$, and $G(\eta)$ – we have $L_\psi(m_\varepsilon, f, \chi) \in \mathbb{Q}(f, \psi, \chi)$. On the other hand, Theorem B2 tells us that we need to add $\Lambda_{k,\psi}$, $G(\psi)$, and $\zeta_\star$ to this field for the algebraicity of $L_\psi(k - \varepsilon, f, \chi)$. Hence if we did use $\mu(\Lambda, k, \psi)$ to define $Y_\psi(m, f, \chi)$, whilst we would still have $Y_\psi(m, f, \chi) \in \mathbb{Q}(f, \psi, \chi)$, we would need to add $\Lambda_{k,\psi}$, $G(\psi)$ and $\zeta_\star$ when thinking about the actual algebraicity of $L_\psi(m, f, \chi)$ itself. This would essentially leave the separation of the two main Theorems B1 and B2 redundant, and yet they deserve to be separated since the algebraicity of Theorem B1 is much stronger than that of B2.

Proof of Theorem B2. Note that $\Omega^{\pm}_{\theta_\chi} = \Omega^{\pm}_{n,k}$. If $m \in \Omega^{+}_{n,k}$ and we are not in cases (**R1**, **R2**) then combine the integral expression of (3.4.1) with Proposition 3.4.3 whereas, if $m \in \Omega^{-}_{n,k}$ (resp. (**R1**, **R2**)), then use the integral expression of (3.4.3) (resp. (3.4.4)) directly. This gives

$$Z_\psi(m, f, \chi) \in c_f(\tau, 1)^{-1} \epsilon_{\mathfrak{y}}(m, \eta) \prod_{p \in \mathbf{b}} g_p\left((\psi^{\mathfrak{c}} \eta^*)(p) p^{-m}\right) \frac{\langle f, K_{\mathcal{S}}(m, \theta) \rangle}{\mu(\Lambda, k, \psi)} \mathbb{Q},$$

which is evidently σ-equivariant over $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Lambda_{k,\psi}, G(\psi), \zeta_\star))$ by Theorem 3.5.2. $\square$

Chapter 4

p-adic L-functions on metaplectic groups

This chapter is mostly adapted from the author's paper [Mer19]; Sect. 4.2.2, which focuses on the case $n = 1$, has been published in [Mer18a].

The p-adic interpolation of special values of L-functions dates back to Kubota-Leopoldt's interpolation of the Riemann zeta function in [KL64], and we noted in the introduction how it was the key insight of Iwasawa that these p-adic L-functions had an algebraic interpretation as ideals of Iwasawa algebras. Subsequent successes of Iwasawa theory to problems such as the BSD conjecture have elevated their role within number theory.

The algebraic theory needed to construct the Iwasawa-theoretic notion of a p-adic L-function is not available for the present setting, but the current analytic theory of metaplectic modular forms allows the p-adic interpolation. We again make use of Shimura's analytic theory, particularly [Shi95b, Shi96, Shi00], to achieve this in this chapter. The p-adic theory of metaplectic modular forms has been studied before, see Ramsey's thesis [Ram06] in the $n = 1$ case; the p-adic L-function for $n = 1$ follows as a result of Shimura's correspondence in [Shi73], nevertheless it was also constructed directly by this author in [Mer18a], some of which is included in this chapter. Note again that there has been decent progress on the algebraic side of metaplectic modular forms, this is the work of Weissman, McNamara, Gan, and Gao that we mentioned in the introduction, with which one can hope to eventually formulate the Iwasawa-theoretic interpretation of the p-adic L-function constructed in this chapter.

The key characteristic of the p-adic L-function is that it has outputs in some p-adic space and interpolates the special values of the complex L-function. They can be defined as the Mellin transform of a p-adic measure (which is a p-adically bounded distribution) and so the task is to construct a p-adic measure interpolating the special values from the previous chapter. The boundedness of this measure is the crux of the proof and the Rankin-Selberg method is employed again to show this.

The p-adic interpolation of L-functions for Siegel modular forms of integral weight has been done before, notably by Panchishkin in Chapters 2 and 3 of [Pan91] for even degrees n and by Böcherer and Schmidt in [BS00] for any n. In this chapter, non-trivial modifications to methods of Panchishkin are made to allow it to work for metaplectic modular forms of any degree.

Following the statement of the main theorem in Section 4.1, we construct the p-stabilisation of a metaplectic modular form in Section 4.2. The p-stabilisation f_0 of a form f is a modification to the level of f so that p now divides it, while the eigenvalues remain largely unchanged and their L-functions are easily related. That p divides the level of f_0 is vital in the following sections, and this process of p-stabilisation means that we get full generality of the level in the main result. For general n this process is abstract and it hinges on Hecke polynomials and the Satake map; for $n = 1$ this stabilisation can be constructed directly via other means, so this is included separately. Due to differences in Hecke theory, this is where key modifications to the methods of Panchishkin are made.

In Section 4.3 we use the trace operator on modular forms to reduce the level of the Rankin-Selberg integral after which, in Sections 4.4 and 4.5 respectively, we give explicit formulae and Fourier developments of theta series and Eisenstein series. The culmination of these sections – the proof of the main theorem – is given in Section 4.6, in which the measures are shown to be p-adically bounded by an application of the abstract Kummer congruences to the Fourier coefficients of Eisenstein series. In [Pan91], Panchishkin uses the Mazur measure interpolating the Dirichlet L-function to show that the Fourier coefficients of the Eisenstein series there satisfy the Kummer congruences. A key change occurs for us in this last step since the Eisenstein series here have rather different Fourier coefficients and we need to show that they satisfy the Kummer congruences from scratch. Strangely, this final step is more immediate for half-integral weights.

This chapter is concluded by a section containing further remarks for the $n = 1$ case. Prior to the research conducted for this chapter we constructed the p-adic L-function for modular forms of half-integral weight in [Mer18a], when $n = 1$. This function was previously known to exist, indirectly, via Shimura's correspondence in [Shi73] between modular forms of half-integral and integral weights, however it had not been established directly and the paper formed a suitable stepping stone for the general $n \geq 1$ case. As alluded to above, the salient difference between the $n = 1$ case and $n \geq 1$ is in the process of p-stabilisation and so we include both cases in full detail here. The rest of the method of [Mer18a] is similar to, and therefore superseded by, the results of this chapter; so we do not detail them fully, but content ourselves with making some remarks on the more minor differences between the two cases.

4.1 Main theorem

Notation. k – half-integral weight; $[k] = k - \frac{1}{2} \in \mathbb{Z}$.
$\mathfrak{b}$ – fractional ideal of $\mathbb{Q}$; $\mathfrak{c}$ – integral ideal of $\mathbb{Z}$.
$(\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}) \subseteq 2\mathbb{Z} \times 2\mathbb{Z}$.
$\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$.
Hecke character $\psi : \mathbb{I}_{\mathbb{Q}}/\mathbb{Q}^\times \to \mathbb{T}$ satisfying (2.1.5) and (2.1.6).
$S^\triangledown$ – set of symmetric half-integral $n \times n$ matrices.
$S^\triangledown_{\mathfrak{b}}$ – set of symmetric $n \times n$ matrices such that $\tau \in N(\mathfrak{b})S^\triangledown$.
ρ_τ – quadratic character associated to $\mathbb{Q}(i^{[n/2]}\sqrt{|2\tau|})/\mathbb{Q}$.
p – an odd prime.

Throughout this chapter, fix a prime $p \neq 2$. Given the general notion of a p-adic L-function being an interpolation of a complex L-function with its values lying in some p-adic space, it is not so clear how to go about defining such a thing. Since the complex L-function takes a variable $s \in \mathbb{C}$ and outputs s-exponents, then there does not appear to be any immediate way to guarantee p-adicity of these outputs. This initial step requires the non-trivial observation that any $s \in \mathbb{C}$ can equally be viewed as a continuous character $\mathbb{R}_{>0} \to \mathbb{C}^\times$ given by $y \mapsto y^s$. Then a complex L-function can equally be thought of as a Mellin transform of such a function. Given some $h : \mathbb{R}_{>0} \to \mathbb{C}^\times$, its *Mellin transform* is the function

$$L_h(s) := \int_{\mathbb{R}_{>0}} h(y)y^s \frac{dy}{y},$$

assuming that h also satisfies appropriate growth conditions, and this is all well and good since we know how to integrate over $\mathbb{R}_{>0}$ with the proto-measure dy. For example, suppose $n = 1$ and f is some classical integer-weight cusp form. Then putting $h(y) = f(iy)$ in the Mellin transform yields the classical L-function

$$\frac{\Gamma(s)}{(2\pi)^s} \sum_{n=1}^{\infty} c_f(n,1)n^{-s} = \int_0^\infty f(iy)y^s \frac{dy}{y},$$

whenever this is convergent, see [Pan91, p. 23] for the details.

Now it is a little clearer how to proceed. The theory of p-adic integration is well established using the notion of p-adic measures. Then one can make a straightforward generalised definition of the Mellin transform in the p-adic setting, after which define the p-adic L-function as the p-adic Mellin transform of characters on $\mathbb{Z}_p^\times$ with respect to a p-adic measure, both of which (the character and measure) have values in the desired p-adic space. Put like this the task seems almost trivial, but recall that we want to *interpolate* the complex L-function at special values, hence we need this measure to interpolate these values too. That such an object is a measure at all, i.e. that it is p-adically bounded, is the crux of the matter.

The algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ is not complete, so denote the completion by $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}}_p$, which space will form the output space of our p-adic L-function. The p-adic norm of $\mathbb{Q}_p$

naturally extends to $\mathbb{C}_p$ and its ring of integers is given by

$$\mathcal{O}_p := \{x \in \mathbb{C}_p \mid |x|_p \leq 1\}.$$

A $\mathbb{C}_p$-analytic function on $W \subseteq \mathbb{C}_p$ is any function $F : W \to \mathbb{C}_p$ that can be written as a power series

$$F(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

convergent in some neighbourhood of x_0 for all $x_0 \in W$, and where $a_n \in \mathbb{C}_p$. Early examples of such functions arose out of interpolating functions on integers, see [Pan91, pp. 12–13], so the question of whether L-functions can be p-adically interpolated is natural.

The domain of the p-adic L-function will be

$$X_p := \{x \in \mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times) \mid x \text{ is continuous}\}.$$

The discussion in [Pan91, pp. 23–25] concerning a natural decomposition of X_p tells us that any $\mathbb{C}_p$-analytic function F on X_p is uniquely determined by its values $F(\chi_0\chi)$ for a fixed $\chi_0 \in X_p$ and χ ranging over non-trivial elements of X_p^{tors}. This torsion subgroup can be identified as the group of primitive Dirichlet characters having p-power conductor.

Of particular interest to us will be the cyclotomic characters

$$\begin{aligned} x_p^{[m]} : \mathbb{Z}_p^\times &\to \mathbb{C}_p^\times \\ y &\mapsto y^{[m]}, \end{aligned}$$

where $m \in \frac{1}{2}\mathbb{Z}$ is such that $[m] = m - \frac{1}{2} \in \mathbb{Z}$. The p-adic L-function will be a $\mathbb{C}_p$-analytic function, hence by the above discussion it will be enough to give the values of the measure on $\chi x_p^{[m]}$ where χ ranges over non-trivial primitive Dirichlet characters of p-power conductor, and m ranges over a certain set of half-integers which is determined by the previous chapter. The measure of $\chi x_p^{[m]}$ will be defined in terms of the values $L_\psi(m, f, \chi)$.

Definition 4.1.1. Let $LC(\mathbb{Z}_p^\times, \mathbb{C}_p)$ denote the $\mathbb{C}_p$-module of all locally constant functions $\mathbb{Z}_p^\times \to \mathbb{C}_p$, and let A be a $\mathbb{C}_p$-module. An A-valued *distribution* on $\mathbb{Z}_p^\times$ is an A-linear homomorphism

$$\nu : LC(\mathbb{Z}_p^\times, \mathbb{C}_p) \to A,$$

which we denote by

$$\nu(\phi) = \int_{\mathbb{Z}_p^\times} \phi d\nu,$$

for any $\phi \in LC(\mathbb{Z}_p^\times, \mathbb{C}_p)$.

When $A = \mathbb{C}$ these are called *complex distributions*, whereas when $A = \mathbb{C}_p$ they are *p-adic distributions.*

The group $\mathbb{Z}_p^\times$ is profinite and we have

$$\mathbb{Z}_p^\times = \varprojlim (\mathbb{Z}/p^i\mathbb{Z})^\times$$

with respect to the natural projections $\pi_{ij} : (\mathbb{Z}/p^i\mathbb{Z})^\times \to (\mathbb{Z}/p^j\mathbb{Z})^\times$, $i \geq j \in \mathbb{Z}$. These projections satisfy the universal property which is the existence, for each $1 \leq i \in \mathbb{Z}$, of a unique projection $\pi_i : \mathbb{Z}_p^\times \to (\mathbb{Z}/p^i\mathbb{Z})^\times$ satisfying $\pi_{ij} \circ \pi_i = \pi_j$ for any $i \geq j$. To any distribution are then associated, by universal projection, a system of functions

$$\nu_i : (\mathbb{Z}/p^i\mathbb{Z})^\times \to A,$$

which satisfy

$$\nu_j(y) = \sum_{x \in \pi_{ij}^{-1}(y)} \nu_i(x) \tag{4.1.1}$$

for any $y \in (\mathbb{Z}/p^j\mathbb{Z}_p)^\times$. To see how this association works, note that each $\phi \in LC(\mathbb{Z}_p^\times, \mathbb{C}_p)$ factors through some $(\mathbb{Z}/p^i\mathbb{Z})^\times$. The distribution ν and the system of functions $\{\nu_i\}_i$ are related by

$$\int_{\mathbb{Z}_p^\times} \phi d\nu = \sum_{x \in \mathbb{Z}/p^i\mathbb{Z}} \phi(x)\nu_i(x),$$

which latter equation, by the relation of (4.1.1) above, makes sense.

Since we shall be working with Dirichlet characters factoring through $(\mathbb{Z}/p^\ell\mathbb{Z})^\times$, we need a way of going backwards from the system $\{\nu_i\}_i$ to an overall distribution on $\mathbb{Z}_p^\times$. The following criterion, from [Pan91, p. 17], tells us when we can do this.

Proposition 4.1.2 (Compatibility criterion)**.** *Consider an arbitrary system of functions*

$$\{\nu_i : (\mathbb{Z}/p^i\mathbb{Z})^\times \to A\}_{i=1}^\infty.$$

If, for any fixed $j \in \mathbb{Z}$ and any function $\phi_j \in LC((\mathbb{Z}/p^j\mathbb{Z})^\times, \mathbb{C}_p)$, the sum

$$\sum_{y \in (\mathbb{Z}/p^i\mathbb{Z})^\times} \phi_j(\pi_{ij}(y))\nu_i(y)$$

is independent of i for large enough $i \geq j$, then there exists a distribution ν on $\mathbb{Z}_p^\times$ associated to $\{\nu_i\}_i$.

Definition 4.1.3. Let $\mathcal{C}(\mathbb{Z}_p^\times, \mathbb{C}_p)$ denote the topological $\mathbb{C}_p$-module of all continuous functions $\mathbb{Z}_p^\times \to \mathbb{C}_p$. A *p-adic measure* is a $\mathbb{C}_p$-module homomorphism

$$\nu : \mathcal{C}(\mathbb{Z}_p^\times, \mathbb{C}_p) \to \mathbb{C}_p.$$

Since $LC(\mathbb{Z}_p^\times, \mathbb{C}_p)$ is dense in $\mathcal{C}(\mathbb{Z}_p^\times, \mathbb{C}_p)$ then any measure ν is uniquely determined by its corresponding distribution – this is just ν restricted to $LC(\mathbb{Z}_p^\times, \mathbb{C}_p)$. The following proposition gives a well-known criterion for when we can run this backwards.

Proposition 4.1.4. *Let ν be a distribution on $\mathbb{Z}_p^\times$. For any $\phi = \lim_{m\to\infty} \phi_m \in \mathcal{C}(\mathbb{Z}_p^\times, \mathbb{C}_p)$, with each $\phi_m \in LC(\mathbb{Z}_p^\times, \mathbb{C}_p)$, define $\nu(\phi) = \lim_{m\to\infty} \nu(\phi_m)$, with all limits taken with respect to the p-adic norm on $\mathbb{C}_p$. Then ν is a measure on $\mathbb{Z}_p^\times$ if and only if there exists a constant $B > 0$ such that*

$$|\nu_i(x)|_p \leq B,$$

for all i and all $x \in (\mathbb{Z}/p^i\mathbb{Z})^\times$.

Proof. The proposition follows from the observation that

$$\left|\int_{\mathbb{Z}_p^\times} \phi d\nu\right|_p = \lim_{m\to\infty}\left|\int_{\mathbb{Z}_p^\times} \phi_m d\nu\right|_p \leq \lim_{m\to\infty} \max_{x\in\mathbb{Z}/p^{i_m}\mathbb{Z}} \{|\nu_{i_m}(x)|_p\},$$

using the non-Archimedean property of the p-adic norm, and the fact that each ϕ_m is locally constant and factors through some $(\mathbb{Z}/p^{i_m}\mathbb{Z})^\times$. Therefore this limit exists if and only if the bound in the proposition holds. □

The final criterion of this section is known as the abstract Kummer congruences. This important and well-known criterion is very general and is due to Katz in [Kat78, p. 258]; we give a specialisation of it. Generally it tells us when ν is a measure, given knowledge of some of the values of its underlying distribution, and so it will be fundamental for our uses.

Proposition 4.1.5 (Kummer Congruences). *Let I be some indexing set and suppose that $\{f_i\}_{i\in I} \subseteq \mathcal{C}(\mathbb{Z}_p^\times, \mathcal{O}_p)$ is such that $\mathrm{span}_{\mathbb{C}_p}\{f_i \mid i \in I\}$ is dense in $\mathcal{C}(\mathbb{Z}_p^\times, \mathbb{C}_p)$. For a given system $\{a_i\}_{i\in I} \subseteq \mathcal{O}_p$ there exists an $\mathcal{O}_p$-module homomorphism $\nu : \mathcal{C}(\mathbb{Z}_p^\times, \mathcal{O}_p) \to \mathcal{O}_p$ such that*

$$\int_{\mathbb{Z}_p^\times} f_i d\nu = a_i$$

if and only if, for any finite subset $S \subseteq I$ and any system $\{b_i\}_{i\in S} \subseteq \mathbb{C}_p$, the condition

$$\sum_{i\in S} b_i f_i \subseteq p^N \mathcal{O}_p$$

implies that

$$\sum_{i\in S} b_i a_i \in p^N \mathcal{O}_p.$$

Note that the Kummer congruences give an $\mathcal{O}_p$-valued measure ν, but it also covers our p-adic measures since a $\mathbb{C}_p$-valued measure can be turned into a $\mathcal{O}_p$-valued one via multiplication by some non-zero constant. The proof of the above can be found in [Pan91, pp. 19–20].

Essentially, the Kummer congruences say that if, for some integer N, any finite linear combination of the integrands is in $p^N\mathcal{O}_p$, then there exists such a measure so long as the corresponding linear combination of the integrals is also in $p^N\mathcal{O}_p$; in other words, so long as the process of p-adic integration "respects" congruences.

If ν is a p-adic measure with $\nu(\chi x_p^m) = a_m(\chi)$ for some $a_m : \{\mathbb{T}\text{-valued characters}\} \to \mathbb{C}_p$ and ω is a primitive $\mathbb{T}$-valued character whose conductor is prime to p, then the twist of ν by ω is given by $[\nu \otimes \omega](\chi x_p^m) = a_m(\chi\omega)$, and this also defines a p-adic measure.

Definition 4.1.6. Let ν be a p-adic measure on $\mathbb{Z}_p^\times$. The *p-adic Mellin transform* of ν is given by

$$L_\nu(x) = \int_{\mathbb{Z}_p^\times} x d\nu,$$

for any $x \in X_p$.

This section is concluded by the statement of the main theorem of this chapter, which gives the desired existence of the p-adic measures, and therefore the p-adic L-functions themselves. Note that we produce two measures and two p-adic L-functions according to the two sets of special values, $\Omega^-_{n,k}$ and $\Omega^+_{n,k}$.

Fix an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. If $\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}] \leq \Gamma[2,2]$ and $f \in \mathcal{S}_k(\Gamma, \psi)$ is a non-zero Hecke eigenform, then we say it is *p-ordinary* if

$$\left| p^{\frac{n(n+1)}{2}} \lambda_{p,1} \cdots \lambda_{p,n} \right|_p = 1,$$

and assume that $p \nmid \mathfrak{c}$. To any such form we associate, in the next section, the p-stabilisation f_0 of f. There is no guarantee that $f_0 \neq 0$, so the assumption that there exists $c_{f_0}(\tau, 1) \neq 0$ in the following theorem in crucial. For a fixed τ recall $\mathfrak{t}$ as the integral ideal such that $h^T (2\tau)^{-1} h \in 4\mathfrak{t}^{-1}$ for all $h \in \mathbb{Z}^n$, and recall the definition of $G_n(\chi)$ from (2.1.4).

Theorem C. *Let $k > 2n$ and let $f \in \mathcal{S}_k(\Gamma, \psi)$ be a p-ordinary Hecke eigenform, where p is any prime such that $p \nmid \mathfrak{c}$. Assume the existence of $0 < \tau \in S_{\mathfrak{b}}^{\triangledown}$ such that $c_f(\tau, 1) \neq 0$ and $c_{f_0}(\tau, 1) \neq 0$. There exist bounded $\mathbb{C}_p$-analytic functions*

$$\nu_f^+ : X_p \to \mathbb{C}_p,$$
$$\nu_f^- : X_p \to \mathbb{C}_p,$$

that are uniquely determined by the following. In both cases $\chi \in X_p^{\mathrm{tors}}$ is a primitive Dirichlet character of conductor p^{ℓ_χ}, with $1 \leq \ell_\chi \in \mathbb{Z}$; set $\eta := \psi\bar{\chi}\rho_\tau$; choose $\mu \in \{0, 1\}$ so that $(\psi\chi)_\infty(-1) = (-1)^{[k]+\mu}$; define $\hat{\tau} := N(\mathfrak{t})(2\tau)^{-1} \in M_n(\mathbb{Z})$; set $\Lambda_\tau(s) := (\Lambda_{\mathfrak{c}}/\Lambda_{\mathfrak{tc}})(\frac{2s-n}{4})$, which is a finite product of Euler factors over $p \mid \mathfrak{t}$ twisted by η, defined by (2.4.1) and (2.4.3), and

$$\mathcal{G}_\tau(s) := \prod_{q \in \mathbf{b}} g_q \left((\psi^{\mathfrak{c}p}\bar{\chi})(q) q^{-s} \right)^{-1},$$

which is a product of polynomials as given in Section 2.4.1; recall the numbers c_m defined in (3.1.3), we have $c_m = n(k+m-n)$; and let

$$d := \begin{cases} \frac{n^2}{2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{4.1.2}$$

(i) For any $m \in \frac{1}{2}\mathbb{Z}$ such that $m - \frac{1}{2} \in \mathbb{Z}$ and $n < m \leq k - \mu$, the measure ν_f^+ is given by

$$\int_{\mathbb{Z}_p^\times} \chi x_p^{[m]} d\nu_f^+ = \iota_p \left[\frac{(-1)^{n[k]} |2\tau|^{\frac{n}{2}+\mu}}{i^d N(\mathfrak{tbc})^{n\mu}} \left| -\frac{N(\sqrt{\mathfrak{tbc}})^2}{2} \hat{\tau} \right|^{-\frac{k+m-\mu-1-2n}{2}} \right.$$
$$\left. \times \frac{p^{n\ell_\chi(n+1-k-m)} G_n(\bar{\chi})}{\Lambda_\tau(m) \mathcal{G}_\tau(m)} \left(p^{\frac{n(n+1)}{2}} \lambda_{p,1} \cdots \lambda_{p,n} \right)^{-\ell_\chi} \frac{L_\psi(m, f, \bar{\chi})}{\pi^{c_m} \langle f, f \rangle} \right],$$

whenever $[m] \equiv [k] + \mu \pmod 2$ (i.e. whenever $m \in \Omega^+_{n,k}$ of Theorem B2) and

$m \neq n + \frac{1}{2}$ (with the further condition that $m \neq n + \frac{3}{2}$ if $(\psi^*\chi)^2 = 1$ and $n > 1$), otherwise the integral is zero.

(ii) For any $m \in \frac{1}{2}\mathbb{Z}$ such that $m - \frac{1}{2} \in \mathbb{Z}$ and $2n + 1 - k + \mu \leq m \leq n$, the measure ν_f^- is given by

$$\int_{\mathbb{Z}_p^\times} \chi x_p^{[m]} d\nu_f^- = \iota_p \left[\frac{(-1)^{n[k]} |2\tau|^{\frac{n}{2}+\mu}}{i^d N(\mathfrak{t}\mathfrak{b}\mathfrak{c})^{n\mu}} \left| -\frac{N(\sqrt{\mathfrak{t}\mathfrak{b}\mathfrak{c}})^2}{2} \hat{\tau} \right|^{-\frac{k+3m-\mu-2-4n}{2}} \right.$$
$$\left. \times \frac{p^{n\ell_\chi(n+1-k-m)} G_n(\bar{\chi})}{\Lambda_\tau(m) \mathcal{G}_\tau(m)} \left(p^{\frac{n(n+1)}{2}} \lambda_{p,1} \cdots \lambda_{p,n} \right)^{-\ell_\chi} \frac{L_\psi(m, f, \bar{\chi})}{\pi^{c_m} \langle f, f \rangle} \right],$$

whenever $[m] \equiv \mu + 1 - [k] \pmod 2$ *(i.e. whenever $m \in \Omega_{n,k}^-$ of Theorem B2), otherwise the integral is zero.*

Remark 4.1.7. The condition that $m \neq n + \frac{1}{2}$, and $m \neq n + \frac{3}{2}$ in certain cases, arises from complications in the Fourier development of the Eisenstein series, for which values there exist non-zero coefficients at $\tau \in S_+$ such that $|\tau| = 0$. This condition should be removable since, by the formulae of Proposition 17.6 in [Shi00], one can interpolate the coefficients at $|\tau| = 0$ using the Mazur measure, as is done in the integral-weight case of [Pan91, pp. 113–114].

Definition 4.1.8. Let $f \in \mathcal{S}_k(\Gamma, \psi)$ be a p-ordinary eigenform and let $\nu_f^\pm$ be the measures of Theorem C. Then one can define two p-adic L-functions of f by the p-adic Mellin transform as follows. If $s \in \mathbb{C}$ is a variable and χ is a primitive Dirichlet character of p-power conductor then put

$$\mathcal{L}_p^\pm(s, f, \chi) := L_{\nu_f^\pm}\left(\chi x_p^{s-\frac{1}{2}}\right) = \int_{\mathbb{Z}_p^\times} \chi x_p^{s-\frac{1}{2}} d\nu_f^\pm,$$

which, by the formulae of Theorem C (i) and (ii), interpolate $L_\psi(s, f, \bar{\chi})$ at the values $m \in \Omega_{n,k}^+ \backslash \{n + \frac{1}{2}\}$ (and $m \neq n + \frac{3}{2}$ if $(\psi^*\chi)^2 = 1$ and $n > 1$) and $m \in \Omega_{n,k}^-$ respectively.

4.2 p-stabilisation

Notation. k – half-integral weight; $[k] = k - \frac{1}{2} \in \mathbb{Z}$.
$\mathfrak{b}$ – fractional ideal of $\mathbb{Q}$; $\mathfrak{c}$ – integral ideal of $\mathbb{Z}$.
$(\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}) \subseteq 2\mathbb{Z} \times 2\mathbb{Z}$.
$P = \{\alpha \in Sp_n(\mathbb{Q}) \mid c_\alpha = 0\}$; $r_P : P_\mathbb{A} \to M_\mathbb{A}$ – lift.
$D = D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$.
$\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$.
Hecke character $\psi : \mathbb{I}_\mathbb{Q}/\mathbb{Q}^\times \to \mathbb{T}$ satisfying (2.1.5) and (2.1.6).
p – an odd prime.
$\varepsilon_b = 1$ if $b \equiv 1 \pmod 4$, $\varepsilon_b = i$ if $b \equiv 3 \pmod 4$ (b is an odd integer).
$\tilde{q} = (q^T)^{-1}$ for any invertible matrix q.

The abstract definition of the p-stabilisation, f_0, of f can be achieved through the factorisation of some Hecke polynomials, which method will be explored in the first part of this section. On the other hand, if $n = 1$ then there are enough concrete results on the action of the various operators involved to allow a direct and explicit construction of f_0, which will be done in the latter half of this section. It will be checked that this explicit construction agrees with the abstract one. In fact, it is pretty much the same method as the abstract one, but done in reverse.

4.2.1 General case

The general method here is adapted from [Pan91, pp. 42–50], in which Panchishkin uses results of Andrianov from Chapter 2 of [And79] to construct the p-stabilisation of integral-weight forms by factorising a certain polynomial and making use of the Satake map.

The polynomial used in [And79] and [Pan91] is the spinor polynomial; for our case we need to use a variation on the symmetric square polynomial. With $p \nmid \mathfrak{c}$, this particular Hecke polynomial is an element $\widetilde{R}_n \in \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}][z]$ of degree 2^n defined by

$$\widetilde{R}_n(x_1, \ldots, x_n; z) = \widetilde{R}_n(z) := \prod_{\delta_i \in \{\pm 1\}} (1 - p^{\frac{n(n+1)}{2}} x_1^{\delta_1} \cdots x_n^{\delta_n} z).$$

This polynomial has an immediate decomposition of the form

$$\widetilde{R}_n(z) = \sum_{m=0}^{2^n} (-1)^m \widetilde{T}_m z^m,$$

where the coefficients $\widetilde{T}_m = \widetilde{T}_m(x_1, \ldots, x_n) \in \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$. By definition of $\widetilde{R}_n$, these coefficients are invariant under the group of Weyl transformations, which is generated by the transformations $x_i \mapsto x_i^{-1}, x_j \mapsto x_j$ for $j \neq i$. Therefore, by the Satake isomorphism ω_p, there exists a polynomial

$$R_n(z) = \sum_{m=0}^{2^n} (-1)^m T_m z^m, \tag{4.2.1}$$

whose coefficients $T_m \in \mathcal{R}_{0p}$ satisfy $\widetilde{T}_m(x_1, \ldots, x_n) = \omega_p(T_m)$. If $n = 1$ then notice from (2.2.10) that we have $\omega_p(A(p)) = \omega_p(T_1) = px_1 + px_1^{-1}$, and therefore $T_1 = A(p)$ is the pth Hecke operator.

To make the methods of Panchishkin and Andrianov work we need to use a different, but closely related, Hecke ring than we have done previously. Let $K = D[\mathfrak{b}^{-1}, \mathfrak{bc}] \cap P_{\mathbb{A}}$, $K[2,2] = D[2,2] \cap P_{\mathbb{A}}$, and put $\Gamma_P = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}] \cap P$. We define

$$\begin{aligned} W_0 &:= \left\{ \operatorname{diag}[\tilde{r}, r] \,\middle|\, r \in \prod_p GL_n(\mathbb{Q}_p \cap \mathbb{Q}) \right\}, \\ W &:= K[2,2] W_0 K[2,2], \end{aligned}$$

and metaplectic lifts

$$\mathfrak{K}[2,2] := \operatorname{pr}^{-1}(K[2,2]), \qquad \mathfrak{K} := \{\alpha \in \mathfrak{K}[2,2] \mid \operatorname{pr}(\alpha) \in G_{\mathbf{f}} \cap K\},$$

$$\mathfrak{W} := \mathrm{pr}^{-1}(W), \qquad \mathfrak{W}_0 := \{\alpha \in \mathfrak{W} \mid \mathrm{pr}(\alpha) \in G_{\mathbf{f}} \cap KW_0K\},$$
$$\widehat{\mathfrak{W}}_0 := \{(\alpha, t) \mid t \in \mathbb{T}, \alpha \in \mathfrak{W}_0\}, \qquad \widehat{\mathfrak{K}} := \{(\alpha, 1) \in \widehat{\mathfrak{W}}_0 \mid \alpha \in \mathfrak{K}\}.$$

Now define the Hecke ring $\mathcal{S} := \mathcal{R}(\widehat{\mathfrak{K}}, \widehat{\mathfrak{W}}_0)$, which differs slightly from that of previous chapters in allowing denominators of p into the elements r and is analogous to the Hecke ring L_0 of [And79, pp. 81–82]. The following three conditions hold: $\widehat{\mathfrak{K}} \subseteq \widehat{\mathfrak{D}}$; $\widehat{\mathfrak{D}}\widehat{\mathfrak{W}}_0 = \widehat{\mathfrak{Z}}_0$; $(\widehat{\mathfrak{D}} \cap \widehat{\mathfrak{W}}_0)\widehat{\mathfrak{W}}_0^{-1} \subseteq \widehat{\mathfrak{K}}$. The first and third of these are trivial, and the second can be seen by multiplying out cosets. Therefore, by Lemma 1.1.3 of [And79] there exists an embedding $\varepsilon : \mathcal{R}(\widehat{\mathfrak{D}}, \widehat{\mathfrak{Z}}_0) \to \mathcal{S}$ defined on single cosets as follows:

$$\varepsilon\left(\bigsqcup_g Dg\right) = \bigsqcup_g Kg. \tag{4.2.2}$$

Let $\mathcal{S}_0$ denote the factor ring of $\mathcal{S}$ formed in the same way as in (2.2.6) but with $\mathfrak{K}$ in place of $\mathfrak{D}$ and $\mathfrak{W}$ in place of $\mathfrak{Z}$. Denote by $A_r \in \mathcal{S}_0$ the element corresponding to $\mathrm{diag}[\tilde{r}, r]$ with $r \in \prod_p GL_n(\mathbb{Q}_p \cap \mathbb{Q})$, and as usual we let $\mathcal{S}_{0p}$ denote the subspace generated by all A_r with $r \in GL_n(\mathbb{Q}_p \cap \mathbb{Q})$. Since any coset of $K_p \backslash K_p \,\mathrm{diag}[\tilde{r}, r]K_p$, for any prime p, has a representative in $\mathrm{diag}[\tilde{r}, r]K_p$ then it has a decomposition of the form found in (2.2.7), namely

$$K_p \,\mathrm{diag}[\tilde{r}, r]K_p = \bigsqcup_{x\in X} \bigsqcup_{s\in Y_x} \bigsqcup_{d\in R_x} K_p\alpha_{d,s}, \qquad \alpha_{d,s} = \begin{pmatrix} \tilde{d} & \tilde{d}s \\ 0 & d \end{pmatrix}, \tag{4.2.3}$$

with $X \subseteq GL_n(\mathbb{Q}_p)$, $R_x \subseteq x\mathcal{O}_p$ representing $\mathcal{O}_p \backslash \mathcal{O}_p x \mathcal{O}_p$, and $Y_x \subseteq S_p$. By the same process seen in Section 2.2, one can define a Satake map $\omega'_p := \omega_{0p} \circ \Phi'_p : \mathcal{S}_{0p} \to \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$ by defining $\Phi'_p : \mathcal{S}_{0p} \to \mathcal{R}(\mathcal{O}_p, GL_n(\mathbb{Q}_p))$ analogously and keeping ω_{0p} the same. The map Φ'_p may no longer necessarily be injective. Let $\varepsilon_{0p} : \mathcal{R}_{0p} \to \mathcal{S}_{0p}$ denote the local embedding obtained from (4.2.2) and notice that $\omega_p = \omega'_p \circ \varepsilon_{0p}$.

There exists an element $U_p \in \mathcal{S}_{0p}$, called the *Frobenius* element, defined by

$$U_p = \Gamma_P \begin{pmatrix} p^{-1}I_n & 0 \\ 0 & pI_n \end{pmatrix} \Gamma_P = \bigsqcup_{u \in S(\mathfrak{b}^{-1}/p^2\mathfrak{b}^{-1})} \Gamma_P \begin{pmatrix} p^{-1}I_n & p^{-1}u \\ 0 & pI_n \end{pmatrix}.$$

If $n = 1$ and $p \mid \mathfrak{c}$, it is well known that $U_p = A_\psi(p)$ is the pth Hecke operator (they both shift the coefficients by p^2, see Proposition 1.5 of [Shi73] or the next subsection); for general $n > 1$, this is no longer true. By definition we have $\omega'_p(U_p) = p^{\frac{n(n+1)}{2}} x_1 \cdots x_n$.

Let $\mathcal{C} := \{A \in \mathcal{S}_{0p} \mid U_pA = AU_p\}$ denote the centraliser of U_p in $\mathcal{S}_{0p}$.

Proposition 4.2.1. *Any $A \in \mathcal{C}$ is a linear combination of double cosets*

$$K_p \begin{pmatrix} \tilde{r} & 0 \\ 0 & r \end{pmatrix} K_p,$$

where $r \in M_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p)$.

Proof. This is the second statement of Proposition 2.1.1 of [And79], with $\delta = 0$ (in the notation of Andrianov). To prove it use $U_p^- := \Gamma_P \left(\begin{smallmatrix} pI_n & 0 \\ 0 & p^{-1}I_n \end{smallmatrix}\right) \Gamma_P$ and multiply out the cosets on the left-hand and right-hand sides of the condition $U_p^- A_r = A_r U_p^-$ to see that, in this case, no entries of r can be p-integral. To finish use the involution $A_r^\iota := A_{r^{-1}}$, which satisfies $U_p^\iota = U_p^-$. $\square$

Proposition 4.2.2. *The map Φ_p' is injective when restricted to $\mathcal{C}$.*

Proof. By the previous proposition, if $A_r \in \mathcal{C}$ then $r \in M_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Z}_p)$. It therefore has the decomposition

$$K_p \operatorname{diag}[\tilde{r}, r] K_p = \bigsqcup_{d,s} K_p \alpha_{d,s},$$

where $\alpha_{d,s}$ is as in (4.2.3) above, $d \in \mathcal{O}_p \backslash \mathcal{O}_p r \mathcal{O}_p$ and $s \in S(\mathfrak{b}^{-1})_p / d^T S(\mathfrak{b}^{-1})_p d$. This can be seen by multiplying out $\operatorname{diag}[\tilde{r}, r] K_p$ and is analogous to the case $p \mid \mathfrak{c}$ in [Shi94, Lemma 2.3]. Now $J(r_P(\alpha_{d,s})) = 1$ by Lemma 2.4 of [Shi95b], and therefore we have $\Phi_p'(A_r) = |\det(r)|_{\mathbb{A}}^{-n-1} \mathcal{O}_p r \mathcal{O}_p$, which shows injectivity. $\square$

The Hecke polynomial $R_n(z)$ has a factorisation given in terms of the U_p and the T_m, which we now give.

Lemma 4.2.3. *With T_m and U_p defined as above, we have*

$$\sum_{m=0}^{2^n} (-1)^m T_m U_p^{2^n - m} = 0.$$

Proof. Denote the sum on the left-hand side by Y, this is in $\mathcal{S}_{0p}$. By an elementary calculation $\widetilde{R}_n(z) = (p^{\frac{n(n+1)}{2}} z)^{2^n} \widetilde{R}_n((p^{n(n+1)} z)^{-1})$ and hence, by (4.2.1), we have

$$T_m = p^{n(n+1)(m-2^{n-1})} T_{2^n - m}. \tag{4.2.4}$$

Define the *degree* of $K_p \operatorname{diag}[\tilde{r}, r] K_p \in \mathcal{S}_{0p}$ to be $\delta(A_r)$, where $p^h \operatorname{diag}[\tilde{r}, r] = \left(\begin{smallmatrix} p^{\delta(A_r)}\tilde{r}_0 & 0 \\ 0 & r_0 \end{smallmatrix}\right)$ $r_0 = p^h r$ and h is the smallest integer such that $p^h \tilde{r}, p^h r \in M_n(\mathbb{Z}_p)$, and of an arbitrary element as the maximum of the degrees of all double cosets appearing in it; this is the same notion of degree as in [And79, p. 88]. If $\omega_p'(X) \in \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$, let $d(X)$ be the minimum power of any x_i appearing in $\omega_p'(X)$; for example we have $d(U_p) = 1$, whereas $d(T_j) = -j$. Then through the Satake map we see that $\delta(X) = 2|d(X)|$ and in particular $\delta(T_j) = 2j$. In Andrianov's notation of [And79] we have $pU_p = \Pi_+^2$ and, since our notions of degree coincide exactly, Proposition 2.1.2 of that paper says that $X\Pi_+^\ell$ belongs to the centraliser of Π_+^ℓ, for any $X \in \mathcal{S}_{0p}$, so long as $\ell \geq \delta(X)$. Thus $T_j \Pi_+^{2j}$ is in the centraliser of Π_+ for any $0 \leq j \in \mathbb{Z}$ and, multiplying by p^j, we conclude that $T_j U_p^j \in \mathcal{C}$ and, by (4.2.4) above, that $Y \in \mathcal{C}$ as well.

The map ω_{0p} is always injective and by Proposition 4.2.2 it is therefore enough to show $\omega_p'(Y) = 0$. For this note

$$\omega_p'(Y) = \omega_p'(U_p)^{2^n} \sum_{m=0}^{2^n} (-1)^m \widetilde{T}_m \cdot (\omega_p'(U_p)^{-1})^m$$

$$= \omega_p'(U_p)^{2^n} \widetilde{R}_n(x_1, \dots, x_n; \omega_p'(U_p)^{-1}),$$

which is zero, since $(1 - \omega_p'(U_p)z) = (1 - p^{\frac{n(n+1)}{2}} x_1 \cdots x_n z)$ is a factor of $\widetilde{R}_n(z)$. $\square$

The above result is similar to that found in [And79, pp. 90–91]. Now, for $0 \leq m \leq 2^n$, define

$$V_{m,p} = V_m := \sum_{\ell=0}^{m} (-1)^\ell T_\ell U_p^{m-\ell} \in \mathcal{S}_{0p}.$$

Proposition 4.2.4. *The Hecke polynomial $R(z)$ can be factorised as*

$$R(z) = \left(\sum_{m=0}^{2^n-1} V_m z^m \right) (1 - U_p z). \tag{4.2.5}$$

Proof. First note that $V_0 = 1$ by definition and that $V_{2^n-1} U_p = -T_{2^n}$ by Lemma 4.2.3. For $1 \leq m \leq 2^n - 2$ we have

$$V_m - V_{m-1} U_p = \sum_{\ell=0}^{m} (-1)^\ell T_\ell U_p^{m-\ell} - \sum_{\ell=0}^{m-1} (-1)^\ell T_\ell U_p^{m-1-\ell} U_p = (-1)^m T_m.$$

Now expanding the right-hand side of the factorisation of (4.2.5) above gives

$$V_0 + \sum_{m=1}^{2^n-2} (V_m - V_{m-1} U_p) z^m - V_{2^n-1} U_p z^{2^n} = \sum_{m=0}^{2^n} (-1)^m T_m z^m = R(z).$$

$\square$

Definition 4.2.5. Let $f \in \mathcal{M}_k(\Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}], \psi)$ be a Hecke eigenform with Satake p-parameters $(\lambda_{p,1}, \dots, \lambda_{p,n})$ for $p \nmid \mathfrak{c}$. Then the *p-stabilisation* of f is denoted by f_0 and is defined by

$$f_0 := \sum_{m=0}^{2^n-1} \left(p^{\frac{n(n+1)}{2}} \lambda_{p,1} \cdots \lambda_{p,n} \right)^{-m} f | V_{m,p}. \tag{4.2.6}$$

Remark 4.2.6. It is not clear from this method that $f_0 \neq 0$ if $f \neq 0$, and therefore the need to assume the existence of τ such that $c_{f_0}(\tau, 1) \neq 0$ in Theorem C. That f_0 may vanish is entirely possible, as is remarked in [Pan91, p. 50]. In [BS00, Section 9], Böcherer and Schmidt give an alternative construction for the p-stabilisation of a Siegel modular form of integral weight, which method does guarantee that $f_0 \neq 0$.

Given an f as in Definition 4.2.5 above recall that $f | A = \widetilde{A}(\lambda_{p,1}, \dots, \lambda_{p,n}) f$, where we have denoted $\widetilde{A} = \omega_p(A) \in \mathbb{C}[x_1^\pm, \dots, x_n^\pm]$.

Proposition 4.2.7. *If $f \in \mathcal{M}_k(\Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}], \psi)$ and $p \nmid \mathfrak{c}$ then $f_0 \in \mathcal{M}_k(\Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}_0], \psi)$, where $\mathfrak{c}_0 := \mathfrak{c} p^{2(2^n-1)}$. Moreover*

$$f_0 | U_p = \left(p^{\frac{n(n+1)}{2}} \lambda_{p,1} \cdots \lambda_{p,n} \right) f_0.$$

Proof. Recall $U_p^- = \Gamma_P \begin{pmatrix} pI_n & 0 \\ 0 & p^{-1} I_n \end{pmatrix} \Gamma_P$, then $f | U_p^- = p^{nk} f(p^2 z)$ has level $\Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}p^2]$ and therefore, as operators, $U_p^- \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}p^2] = U_p^-$. As before, let ι be the involution on

$\mathcal{S}_{0p}$ defined by $A_r^\iota = A_{r^{-1}}$. The elements U_p^- and U_p satisfy $U_p = (U_p^-)^\iota$ and, by the argument found in [Pan91, p. 49], we have $U_p\Gamma[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}p^2] = U_p$ as operators as well. Hence $f|U_p \in \mathcal{M}_k(\Gamma[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}p^2], \psi)$ and the first property follows by definition of f_0 and $V_{m,p}$.

Set $\lambda_0 := p^{\frac{n(n+1)}{2}}\lambda_{p,1}\cdots\lambda_{p,n}$ and define $f|\lambda_0 = \lambda_0 f$. The second property is now given by the following calculation:

$$\begin{aligned} f_0|(\lambda_0 - U_p) &= \lambda_0 f_0|(1 - \lambda_0^{-1}U_p) \\ &= \lambda_0 f \left| \left[\sum_{m=0}^{2^n-1} (\lambda_0^{-1})^m V_{m,p} \right] (1 - \lambda_0^{-1}U_p) \right. \\ &= \lambda_0 f | R_n(\lambda_0^{-1}), \end{aligned}$$

where Proposition 4.2.4 was used in the last line and Definition 4.2.5 in the penultimate. This is zero since $f|R_n(\lambda_0^{-1}) = \widetilde{R}_n(\lambda_{p,1}, \dots, \lambda_{p,n}; \lambda_0^{-1})f$ and $(1 - \lambda_0 z)$ is a factor of $\widetilde{R}_n(\lambda_{p,1}, \dots, \lambda_{p,n}; z)$. ☐

For any other primes $q \neq p$ the operators $V_{m,p}$ and $\varepsilon_q(A(q))$ commute, so f_0 and f share the same eigenvalues away from p and hence the following corollary.

Corollary 4.2.8. *If $1 \leq \ell \in \mathbb{Z}$ and p^ℓ is the conductor of χ, then $L_\psi(s, f, \chi) = L_\psi(s, f_0, \chi)$.*

4.2.2 The case $n = 1$

If $n = 1$ then the classical theory of half-integral weight forms was substantially developed by Shimura in [Shi73]. In this paper, Shimura proves a lot of concrete results regarding the action of Hecke operators, the kind of which one sees when encountering the fundamental theory of classical integer-weight modular forms for the first time. It is shown why we only consider the p^2th Hecke operators (since the pth operator is 0 if $p \nmid \mathfrak{c}$), and the Fourier development of $f|A(p)$ is given explicitly. As we have seen, it is one thing to define the factor of automorphy of weight $\frac{1}{2}$ for $\gamma \in \Gamma[2, 2]$ by using the theta series, but to do so for the action of the Hecke operators is a whole other issue entirely. This is what Shimura first achieves in [Shi73] and is what facilitates the subsequent concrete theory of Hecke operators. That paper is particularly well known for its nexus, which is the surprising result that cusp forms of half-integral weight $\frac{\kappa}{2}$, with $\kappa \in \mathbb{Z}$ odd, are in one-to-one correspondence with cusp forms of integral weight $\kappa - 1$. We dub this the Shimura correspondence, and we explore it in more depth at the end of this section. The key implication of this correspondence for us is that the p-adic L-function for $n = 1$ is already known to exist from the integral-weight case, as this correspondence preserves the L-functions modulo some twisting by quadratic characters. Note that [Shi73] works with congruence subgroups of the form $\Gamma_0(\mathfrak{c}) = \Gamma[1, \mathfrak{c}]$, where $4 \mid \mathfrak{c}$, and the factors of automorphy are obtained via slightly different theta series. Recasting the results of [Shi73] for our setting is fairly simple and has been done by Shimura in the book [Shi12]; for simplicity we assume that $\mathfrak{b} = 2^{-1}\mathbb{Z}$ and thus any $f \in \mathcal{M}_k(\Gamma, \psi)$ has Fourier expansion of the form

$$f(z) = \sum_{n=0}^{\infty} c_f(n) e(nz/2),$$

where $c_f(n) = c(n; f)$ is the coefficient $c_f(\tau/2, 1)$, for $0 \leq \tau \in \mathbb{Z}$, of previous sections.

Let $H(\mathbb{H})$ denote the space of all holomorphic functions $\varphi : \mathbb{H} \to \mathbb{C}$ and define

$$\mathfrak{G}^+ := \{(\alpha, \varphi) \in GL_2^+(\mathbb{Q}) \times H(\mathbb{H}) \mid \varphi(z)^2 \in |\alpha|^{-\frac{1}{2}}(cz + d)\mathbb{T}\},$$

with group law

$$(\alpha, \varphi(z))(\beta, \psi(z)) = (\alpha\beta, \varphi(\beta z)\psi(z)),$$

and then put $\mathfrak{G} := \{(\alpha, \varphi) \in \mathfrak{G}^+ \mid \alpha \in SL_2(\mathbb{Q})\}$. The group $\mathfrak{G}$ is precisely the metaplectic group $Mp_1(\mathbb{Q})$ and is the same $\mathfrak{G}$ appearing at the beginning of Section 2.3.2. If $k = \frac{\kappa}{2}$, where $\kappa \in \mathbb{Z}$ is odd, and $f : \mathbb{H} \to \mathbb{C}$ then we define the action of $\xi = (\alpha, \varphi) \in \mathfrak{G}^+$ on f to be

$$(f||_k\xi)(z) = \varphi(z)^{-\kappa} f(\alpha z). \tag{4.2.7}$$

In this case we have

$$\begin{aligned} h_\sigma(z) &= \varepsilon_{d_\alpha}^{-1} \left(\frac{2c_\alpha}{d_\alpha}\right) (c_\alpha z + d_\alpha)^{\frac{1}{2}}, \\ h_\sigma(z)^2 &= \left(\frac{-4}{d_\alpha}\right) (c_\alpha z + d_\alpha), \end{aligned}$$

for any $\sigma \in G \cap D[2, 2]$ with $\mathrm{pr}(\sigma) = \alpha$. Congruence subgroups of $\mathfrak{G}$ are given as in the start of Section 2.3.2 and for the rest of this section, when we write Γ we are considering its image, $\widehat{\Gamma}$, under the embedding $\gamma \mapsto (\gamma, h_\gamma)$. Hecke operators in this case are considered as double cosets $\Delta\xi\Delta$, where $\Delta \leq \mathfrak{G}$ is a congruence subgroup and $\xi = (\alpha, \varphi) \in \mathfrak{G}^+$. Their action on modular forms is given by decomposing into single cosets and using the slash operator, (4.2.7) above, of $\mathfrak{G}^+$ on modular forms.

The first point of interest is that if $p \nmid \mathfrak{c}$ then the traditional pth Hecke operator, associated to the double coset $\Gamma\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right), p^{\frac{1}{4}}\right)\Gamma$, is 0 as an operator on $\mathcal{M}_k(\Gamma, \psi)$. That the following sequence

$$1 \to \mathbb{T} \to \mathfrak{G} \to SL_2(\mathbb{Q}) \to 1 \tag{4.2.8}$$

is exact, is well known. Thus for some lift $L : SL_2(\mathbb{Q}) \to \mathfrak{G}$ we can write

$$L(\alpha\gamma\alpha^{-1}) = \xi L(\gamma)\xi^{-1}(1, t(\gamma)),$$

where $\xi = (\alpha, \varphi(z)) \in \mathfrak{G}$, $\gamma \in \Gamma \cap \alpha^{-1}\Gamma\alpha$, and $t : \Gamma \cap \alpha^{-1}\Gamma\alpha \to \mathbb{T}$ is a homomorphism. Then Proposition 1.0 of [Shi73] tells us that $f|\Gamma\xi\Gamma = 0$ if t^κ is non-trivial.

Proposition 4.2.9 ([Shi73], pp. 447–448). *Let* $\gamma \in \Gamma[2, 2]$, $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & m \end{smallmatrix}\right)$, $\xi = (\alpha, tm^{-\frac{1}{4}})$ *for* *any $t \in \mathbb{T}$ and $m \in \mathbb{Z}$. Then*

$$\xi\hat{\gamma}\xi^{-1} = \widehat{\alpha\gamma\alpha^{-1}}\left(1, \left(\frac{m}{d_\gamma}\right)\right)$$

if $\gamma \in \Gamma[2,2] \cap \alpha^{-1}\Gamma[2,2]\alpha$.

So given the double coset $\Gamma\xi\Gamma$, where $\xi = \left(\left(\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}\right), p^{\frac{1}{4}}\right)$, the above proposition tells us that $t(\gamma) = \left(\frac{p}{d_\gamma}\right)$. Since κ is odd, we see that t^κ is non-trivial and $f|\Gamma\xi\Gamma = 0$ for any $f \in \mathcal{M}_k(\Gamma, \psi)$. It is precisely the existence of a non-trivial $\ker(\mathrm{pr}) = \mathbb{T}$ in the exact sequence of (4.2.8) that causes the pth Hecke operator to disappear, which kernel naturally does not appear in the integral-weight case. Anyway, curiously, the same is no longer true if $p \mid \mathfrak{c}$.

Proposition 4.2.10 ([Shi73], pp. 448–449)**.** *Let $0 < m \in \mathbb{Z}$ be such that the conductor of $\mathbb{Q}(\sqrt{m})$ divides $\mathfrak{c}$ and the primes of m also divide $\mathfrak{c}$. If $f \in \mathcal{M}_k(\Gamma, \psi)$ and $\xi = \left(\left(\begin{smallmatrix}1 & 0\\ 0 & m\end{smallmatrix}\right), m^{\frac{1}{4}}\right)$ then*

$$f|\Gamma\xi\Gamma = m^{1-\frac{k}{2}} \sum_{n=0}^{\infty} c_f(mn)e(nz/2) \in \mathcal{M}_k\left(\Gamma, \psi \otimes \left(\tfrac{m}{\cdot}\right)\right).$$

With U_p as in the previous subsection, we have $A(p) = U_p$ if $p \mid \mathfrak{c}$, and these correspond to $(p^{-1}I_2, 1)\Gamma\left(\left(\begin{smallmatrix}1 & 0\\ 0 & p^2\end{smallmatrix}\right), \sqrt{p}\right)\Gamma$. By the above proposition the Frobenius element U_p shifts the coefficients of f to the tune of $c(n; f|U_p) = p^{2-k}c_f(p^2n)$.

Let $f \in \mathcal{S}_k(\Gamma)$ and η be a Hecke character of conductor $\mathfrak{f}$. The twist of f by η is given by

$$(f \otimes \eta)(z) := \sum_{n=1}^{\infty} \eta_{\mathfrak{f}}^{-1}(n)c_f(n)e(nz/2),$$

which, since $\mathfrak{b} = 2^{-1}\mathbb{Z}$ and we have $\mathfrak{bc} \subseteq \mathfrak{b}^{-1}$, belongs to $\mathcal{S}_k(\Gamma[\mathfrak{b}^{-1}, \mathfrak{bcf}^2], \eta^2)$. This latter fact can be seen by generalisation of Proposition 17 (b) in [Kob84, pp. 127–128], which requires the assumption $\mathfrak{bc} \subseteq \mathfrak{b}^{-1}$. Let $\eta^{(p)}$ be the Hecke character such that $(\eta^{(p)})^*(n\mathbb{Z}) := \left(\frac{n}{p}\right)$. This has conductor p and define the p-stabilisation in this case by

$$f_1(z) := f(z) - \left(\frac{-1}{p}\right)^{[k]} p^{-\frac{1}{2}}\lambda_{p,1}^{-1}\left(f \otimes \eta^{(p)}\right)(z) - p^{k-1}\lambda_{p,1}^{-1}f(p^2z), \tag{4.2.9}$$

which we can see belongs to $\mathcal{S}_k(\Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}p^2])$ right away. By considering the action, given in [Shi73], of the Hecke operators on the Fourier coefficients of forms we can show directly that f_1 is an eigenform that shares eigenvalues with f for $A(q)$, $q \neq p$, but has eigenvalue $p\lambda_{p,1}$ for $A(p)$.

Corollary 1.8 of [Shi73] tells us that, if $f|A(p) = \Lambda(p)f$ and $p^2 \nmid t$, then

$$\Lambda(p)c_f(t) = p^{2-k}c_f(p^2t) + \left(\frac{-1}{p}\right)^{[k]} \sqrt{p}\left(\frac{t}{p}\right)c_f(t), \tag{4.2.10}$$

$$p^{2-k}c_f(tp^{2m+2}) = \Lambda(p)c_f(tp^{2m}) - p^k c_f(tp^{2m-2}), \tag{4.2.11}$$

for any $0 < m \in \mathbb{Z}$ (in the notation of [Shi73] we have $\omega_p = p^{k-2}\Lambda(p)$).

Proposition 4.2.11. *Suppose $f \in \mathcal{S}_k(\Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}])$ has eigenvalues $\Lambda(p) = \Lambda(A(p))$, and that $p \nmid \mathfrak{c}$. The following three formulae hold:*

$$\left(f \otimes \eta^{(p)}\right)|U_p = 0; \tag{4.2.12}$$

$$f(p^2 z)|U_p = p^{2-k} f; \tag{4.2.13}$$

$$f|U_p = \Lambda(p) f - \left(\frac{-1}{p}\right)^{[k]} \sqrt{p}\left(f \otimes \eta^{(p)}\right) - p^k f(p^2 z). \tag{4.2.14}$$

Proof. That (4.2.12) and (4.2.13) are true is a consequence of $c(n; f|U_p) = p^{2-k} c_f(p^2 n)$.

The formula in (4.2.14) is not so immediate and we need to make use of the coefficient identities preceding the proposition, (4.2.10) and (4.2.11). If $p^2 \nmid n$ then by (4.2.10) we have

$$c(n; \Lambda(p) f - p^k f(p^2 z)) = p^{2-k} c_f(p^2 n) + \left(\frac{-1}{p}\right)^{[k]} \sqrt{p}\left(\frac{n}{p}\right) c_f(n),$$

whereas if $p^2 \mid n$ then by (4.2.11) we have

$$\begin{aligned} c(n; \Lambda(p) f - p^k f(p^2 z)) &= \Lambda(p) c_f(n) - p^k c_f\left(\frac{n}{p^2}\right) \\ &= p^{2-k} c_f(p^2 n). \end{aligned}$$

With this, and the fact that $c\left(n; f \otimes \eta^{(p)}\right) = 0$ for $p^2 \mid n$, we get

$$\Lambda(p) f - p^k f(p^2 z) = f|U_p + \left(\frac{-1}{p}\right)^{[k]} \sqrt{p}\left(f \otimes \eta^{(p)}\right),$$

and hence (4.2.14). □

Proposition 4.2.12. *The form f_1 is an eigenform. If $q \neq p$ then f_1 and f share the same eigenvalues for $A(q)$, whereas $f_1|U_p = p\lambda_{p,1} f_1$.*

Proof. Equations (4.2.12) – (4.2.14) of the previous proposition give

$$f_1|U_p = \Lambda(p) f - \left(\frac{-1}{p}\right)^{[k]} \sqrt{p}\left(f \otimes \eta^{(p)}\right) - p^k f(p^2 z) - p\lambda_{p,1}^{-1} f$$

and by the definition of $\lambda_{p,1}$, given in (2.2.11), we have $\Lambda(p) = p\lambda_{p,1} + p\lambda_{p,1}^{-1}$. This gives

$$\begin{aligned} f_1|U_p &= p\lambda_{p,1} f - \left(\frac{-1}{p}\right)^{[k]} \sqrt{p}\lambda_{p,1}\lambda_{p,1}^{-1}\left(f \otimes \eta^{(p)}\right) - p^k \lambda_{p,1}\lambda_{p,1}^{-1} f(p^2 z) \\ &= p\lambda_{p,1}\left[f - \left(\frac{-1}{p}\right)^{[k]} p^{-\frac{1}{2}}\lambda_{p,1}^{-1}\left(f \otimes \eta^{(p)}\right) - p^{k-1}\lambda_{p,1}^{-1} f(p^2 z)\right] \\ &= p\lambda_{p,1} f_1. \end{aligned}$$

If $q \neq p$ then the commutativity of the coset decompositions, given in [Shi73, p. 451], with $\left(\begin{smallmatrix} p & 0 \\ 0 & p^{-1} \end{smallmatrix}\right)$ show that $f(p^2 z)|A(q) = (f|A(q))(p^2 z)$. By Theorem 1.7 of [Shi73] we have

$$c(n; f|A(q)) = q^{2-k} c_f(nq^2) + \left(\frac{-1}{q}\right)^{[k]} \sqrt{q}\left(\frac{n}{q}\right) c_f(n) + q^k c_f(nq^{-2}),$$

$$c\left(n;\left(f\otimes\eta^{(p)}\right)|A(q)\right)=q^{2-k}\left(\frac{nq^2}{p}\right)c_f(nq^2)+\left(\frac{-1}{q}\right)^{[k]}\sqrt{q}\left(\frac{n}{pq}\right)c_f(n)$$
$$+q^k\left(\frac{nq^{-2}}{p}\right)c_f(nq^{-2}),$$

where we understand $c_f(nq^{-2})=0$ if $q^2\nmid n$. Since $\left(\frac{nq^2}{p}\right)=\left(\frac{n}{p}\right)$ for any n and $\left(\frac{nq^{-2}}{p}\right)=\left(\frac{n}{p}\right)$ if $q^2\mid n$ we get $c\left(n;\left(f\otimes\eta^{(p)}\right)|A(q)\right)=\left(\frac{n}{p}\right)c(n;f|A(q))$, which tells us that $\left(f\otimes\eta^{(p)}\right)|A(q)=(f|A(q))\otimes\eta^{(p)}$. Hence by definition f_1 and f share the same eigenvalue for $A(q)$. □

The above explicit calculations for $n=1$ are actually the same as those found in the general case of the previous section, but run in reverse. It is nice to specialise the definition of f_0 in (4.2.6) to $n=1$ and see how this works out to give us f_1 of (4.2.9). By definition $V_{1,p}=U_p-T_1=U_p-A(p)$ when $n=1$ and so

$$f_0=f+(p\lambda_{p,1})^{-1}f|V_{1,p}=f+p^{-1}\lambda_{p,1}^{-1}f|U_p-p^{-1}\lambda_{p,1}^{-1}\Lambda(p)f$$

which, by (4.2.14), gives

$$f_0=f-\left(\frac{-1}{p}\right)^{[k]}p^{-\frac{1}{2}}\lambda_{p,1}^{-1}\left(f\otimes\eta^{(p)}\right)-p^{k-1}\lambda_{p,1}^{-1}f(p^2z);$$

this is exactly f_1 of (4.2.9).

There is another way one can anticipate the slightly odd form of f_1 given in (4.2.9). At least one can see, by analogy with the integral-weight case, that f_1 should have level $\mathfrak{c}p^2$. The p-stabilisation of an integral-weight form F of level $\Gamma[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}]$, when $p\nmid\mathfrak{c}$, is given by the relatively benign

$$F_0(z):=F(z)-p^{k-1}\lambda_{p,1}^{-1}F(pz),\tag{4.2.15}$$

which is of level $\mathfrak{c}p$. It has eigenvalue $p\lambda_{p,1}$ at p and the same eigenvalues as F elsewhere. The aforementioned Shimura correspondence S_t preserves eigenvalues, and its definition depends on some square-free integer t. The point being here that if we can choose t such that the image $S_t(f_1)$ has level $\mathfrak{c}p$, then $S_t(f_1)$ and $S_t(f)_0$ have precisely the same eigenvalues, so they are multiples of each other. Even better, using the definition of the Shimura correspondence, we can explicitly give this multiple.

Firstly, Proposition 3.7 of [Shi73] tells us what we can take as the level of $S_t(f_1)$. Take $t=2$ and assume $p>3$. For simplicity, we make the further assumptions that $\mathfrak{c}=4\mathfrak{c}'$, where $\mathfrak{c}'$ is odd and square-free, and that $f_1|A(p)\neq 0$ for all $p\mid\mathfrak{c}'$ (other cases are similar, but are messier notationally). Since $f_1\in\mathcal{S}_k(\Gamma[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}p^2])$ has trivial character, the notation of Proposition 3.7 in [Shi73] becomes

$$N=4\mathfrak{c}'p^2,\quad M_t=8,\quad M'=8,\quad H=\mathfrak{c}'p,\quad K_0=\mathfrak{c}'p,\quad N^*=8(\mathfrak{c}')^2p^2,$$

and the proposition says that $N^*/2K_0=\mathfrak{c}p$ can be taken as the level of $S_2(f_1)$. So $S_2(f_1)$

is a constant multiple of $S_2(f)_0$, since the eigenspaces are one dimensional here. The precise statement of the Shimura correspondence, found in [Shi73, p. 458], says that for a half-integral weight eigenform g of trivial character and weight k such that $2k \geq 3$ one has

$$\sum_{n=1}^{\infty} c(n; S_2(g))n^{-s} = \left(\sum_{m=1}^{\infty} \left(\frac{-1}{m}\right)^{[k]} \left(\frac{2}{m}\right) m^{[k]-1-s}\right) \left(\sum_{m=1}^{\infty} c_g(2m^2)m^{-s}\right),$$

which first of all tells us that $c(1; S_2(g)) = c_g(2)$ for any such g. So $c(1; S_2(f_1)) = c_{f_1}(2)$. Notice by the definition of F_0, in (4.2.15) above, that $c(1; S_2(f)_0) = c(1; S_2(f)) = c_f(2)$. Then by the definition of f_1, in (4.2.9), we have got

$$c_{f_1}(2) = \left[1 - \left(\frac{-1}{p}\right)^{[k]} \left(\frac{2}{p}\right) p^{-\frac{1}{2}} \lambda_{p,1}^{-1}\right] c_f(2).$$

Putting all this together gives

$$S_2(f_1) = \left[1 - \left(\frac{-1}{p}\right)^{[k]} \left(\frac{2}{p}\right) p^{-\frac{1}{2}} \lambda_{p,1}^{-1}\right] S_2(f)_0,$$

and hence the integral-weight and half-integral weight processes of p-stabilisation really are different sides of the same coin.

4.3 Tracing the Rankin-Selberg integral

Notation. k – half-integral weight; $[k] = k - \frac{1}{2} \in \mathbb{Z}$.
$\mathfrak{b}$ – fractional ideal of $\mathbb{Q}$; $\mathfrak{c}$ – integral ideal of $\mathbb{Z}$.
$(\mathfrak{b}^{-1}, \mathfrak{bc}) \subseteq 2\mathbb{Z} \times 2\mathbb{Z}$.
$\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}]$.
Hecke character $\psi : \mathbb{I}_{\mathbb{Q}}/\mathbb{Q}^{\times} \to \mathbb{T}$ satisfying (2.1.5) and (2.1.6).
Dirichlet character χ of conductor p^{ℓ_χ}.
$S^{\triangledown}$ – set of symmetric half-integral $n \times n$ matrices.
$S^{\triangledown}_{\mathfrak{b}}$ – set of symmetric $n \times n$ matrices such that $\tau \in N(\mathfrak{b})S^{\triangledown}$.
$\mathsf{X}_p = M_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p)$.
$Z_0 = \{\mathrm{diag}[\tilde{q}, q] \mid q \in GL_n(\mathbb{Q})_{\mathbf{f}} \cap \prod_p \mathsf{X}_p\}$.
$\mathfrak{t}$ – integral ideal such that $h^T(2\tau)^{-1}h \in 4\mathfrak{t}^{-1}$ for all $h \in \mathbb{Z}^n$.
ρ_τ – quadratic character associated to $\mathbb{Q}(i^{[n/2]}\sqrt{|2\tau|})/\mathbb{Q}$.
$\Lambda_\tau(s) = (\Lambda_{\mathfrak{c}}/\Lambda_{\mathfrak{tc}})(\frac{2s-n}{4})$.
$\mathcal{G}_\tau = \prod_{q \in \mathbf{b}} g_q((\psi^{\mathfrak{c}p}\bar{\chi})(q)q^{-s})^{-1}$.
p – an odd prime.
$\mathfrak{c}_0 = \mathfrak{c}p^{2(2^n-1)}$.

With the relationship, established in Corollary 4.2.8, between $L(s, f, \chi)$ and $L(s, f_0, \chi)$ the focus is now shifted to the latter. Once again, we apply the Rankin-Selberg method to it, and in this section the integral expression is modified via use of the trace operator. Recall that the level of the inner product of the Rankin-Selberg integral expression is

$\mathfrak{y} = \mathfrak{c} \cap (4\mathfrak{t}\mathfrak{f}^2)$, where $\mathfrak{f}$ is the conductor of χ. In this section the level of this integral is reduced to eschew any dependence on the character χ.

We now make the assumption that $f_0 \neq 0$ and that we can fix $0 \leq \tau \in S_{\mathfrak{b}}^{\nabla}$ such that $c_{f_0}(\tau, 1) \neq 0$. Define the integral matrix

$$\hat{\tau} : = N(\mathfrak{t})(2\tau)^{-1} \in S(\mathbb{Z}). \tag{4.3.1}$$

Take a Dirichlet character χ of modulus p^ℓ and conductor p^{ℓ_χ} with $0 \leq \ell_\chi \leq \ell \in \mathbb{Z}$, choose a $\mu \in \{0, 1\}$ such that $(\psi_\infty \chi)(-1) = (-1)^{[k]+\mu}$, and put $\eta := \psi\bar{\chi}\rho_\tau$.

Since this section involves many levels, and liftings of modular forms through these levels, we first define and clarify this schematically. Consider $\mathfrak{b}$ fixed and note by Theorem 2.1.5 (i) that $\mathfrak{b}^{-1} \mid \mathfrak{t}$, therefore we can think of f_0 as having level $\Gamma[\mathfrak{b}^{-2}, \mathfrak{b}^2\mathfrak{t}\mathfrak{c}_0]$ where, recall, $\mathfrak{c}_0 = \mathfrak{c}p^{2(2^n-1)}$. Define the integral ideal

$$\mathfrak{y}_\chi := [\mathfrak{t}\mathfrak{c}p^{\ell_\chi}]^2.$$

The ideal $\mathfrak{y}_\chi$ can be taken as the level of the integral in the Rankin-Selberg expression of $L_\psi(s, f_0, \chi)$ only if $\ell_\chi \geq 2^n - 1$; to avoid this condition we choose higher levels. The levels we move through are $\Gamma_\alpha := \Gamma[\mathfrak{b}^{-2}, \mathfrak{b}^2\mathfrak{y}_\alpha]$, indexed by $\alpha \in \{r, \ell \in \mathbb{Z} \mid 0 \leq \ell_\chi \leq \ell \leq r\} \cup \{0\}$, where the $\mathfrak{y}_\alpha$ are defined as follows. The base level, to which everything will be reduced, is $\mathfrak{y}_0$, and we arrange their definitions below in order of divisibility:

$$\begin{array}{l}
\vdots \\
\rotatebox{90}{$\subseteq$} \\
\mathfrak{y}_r := \mathfrak{y}_0 p^{2r} \\
\rotatebox{90}{$\subseteq$} \\
\vdots \\
\rotatebox{90}{$\subseteq$} \\
\mathfrak{y}_\ell := \mathfrak{y}_0 p^{2\ell} \\
\rotatebox{90}{$\subseteq$} \\
\vdots \\
\\
\rotatebox{90}{$\subseteq$} \\
\mathfrak{y}_1 := \mathfrak{y}_0 p^2 \\
\rotatebox{90}{$\subseteq$} \\
\mathfrak{y}_0 := \mathfrak{t}^2\mathfrak{c}\mathfrak{c}_0.
\end{array}$$

Generally, in the Kummer congruences, we shall take a set of Dirichlet characters of varying moduli p^ℓ and consider their respective Rankin-Selberg integrals of varying levels $\mathfrak{y}_\ell$, then take a single $r \geq 0$ so that all characters are all defined modulo p^r, and lift up each integral to $\mathfrak{y}_r$ first from $\mathfrak{y}_\ell$. We then trace the Rankin-Selberg integral back down to $\mathfrak{y}_0$, which process is given in the rest of this section. This is so that we can treat all characters uniformly. In specific cases, i.e. when we consider a single primitive Dirichlet character such that $\ell = \ell_\chi \geq 2^n - 1$, one can use $\mathfrak{y}_\chi$ as the level of the integral in the expression of $L(s, f_0, \chi)$ and we need not lift up to r in the first place; such a case is given as an example at the end of this section but for full generality this will not be used later. The Rankin-Selberg integral of (2.4.4) becomes

$$\begin{aligned} L_\psi(s, f_0, \bar{\chi}) = & \left[\Gamma_n\left(\tfrac{s-n-1+k+\mu}{2}\right) 2c_{f_0}(\tau, 1)\right]^{-1} N(\mathfrak{b})^{n(n+1)} |4\pi\tau|^{\frac{s-n-1+k+\mu}{2}} \\ & \times \Lambda_\tau(s)\mathcal{G}_\tau(s)\left\langle f_0, \theta_{\bar{\chi}}\mathcal{E}_{k-\frac{n}{2}-\mu}(\cdot, \tfrac{2s-n}{4}; \bar{\eta}, \Gamma_r)\right\rangle_{\mathfrak{y}_r} V_r, \end{aligned} \tag{4.3.2}$$

in which $V_r := \mathrm{Vol}(\Gamma_r \backslash \mathbb{H}_n)$, and recall the notation Λ_τ and $\mathcal{G}_\tau$ from Theorem C.

Write $Y_\alpha := N(\mathfrak{b})\sqrt{N(\mathfrak{y}_\alpha)} \in \mathbb{Z}$ for $\alpha \in \{0, \ell, r, \chi\}$; we have $Y_0 = N(\mathfrak{tbc})p^{2^n-1}$, $Y_\ell = Y_0 p^\ell$, and $Y_r = Y_0 p^r$. Notice also that $Y_\chi = Y_0 p^{\ell_\chi - 2^n - 1}$ if $\ell_\chi \geq 2^n - 1$.

If $r \geq 0$, then the trace map $\mathrm{Tr}^{\mathfrak{y}_r}_{\mathfrak{y}_0} : \mathcal{M}_k(\Gamma_r, \psi) \to \mathcal{M}_k(\Gamma_0, \psi)$ is defined in the same way as the trace map from (3.5.1) in the previous chapter, except without the need for the character to appear in the sum defining the output. If $g \in \mathcal{M}_{\frac{n}{2}+\mu}(\Gamma[\mathfrak{b}^{-2}, \mathfrak{b}^2\mathfrak{y}_r], \chi\rho_\tau)$, then set

$$F_g(z, s) := g(z)\mathcal{E}_{k-\frac{n}{2}-\mu}(z, \tfrac{2s-n}{4}; \bar{\eta}, \Gamma_r)$$

and we have

$$\mathrm{Tr}^{\mathfrak{y}_r}_{\mathfrak{y}_0}(F_g) = \sum_{u \in S(\mathbb{Z}/p^{2r}\mathbb{Z})} F_g \Big\|_k \begin{pmatrix} I_n & 0 \\ N(\mathfrak{y}_0)u & I_n \end{pmatrix} = \sum_{u \in S(\mathfrak{b}^{-2}/p^{2r}\mathfrak{b}^{-2})} F_g \Big\|_k \begin{pmatrix} I_n & 0 \\ Y_0^2 u & I_n \end{pmatrix}.$$

Let $M \in \mathbb{Z}$ and define the matrix

$$\iota_M := \begin{pmatrix} 0 & -M^{-1}I_n \\ MI_n & 0 \end{pmatrix}, \tag{4.3.3}$$

which belongs to $P_\mathbb{A}\iota$ and is therefore in $\mathfrak{M}$. We associate to ι_M an operator $W(M)$, acting on modular forms g of weight $\kappa \in \frac{1}{2}\mathbb{Z}$ by $g|W(M) := g||_\kappa \iota_M$.

Proposition 4.3.1. *Let χ be a character modulo p^ℓ and take g and F_g as above. If $r \geq 0$ is an integer then*

$$\langle f_0, F_g(\cdot, s)\rangle_{\mathfrak{y}_r} = (-1)^{n[k]}\langle f_0, H_g(\cdot, s)|U_p^r W(Y_0)\rangle_{\mathfrak{y}_0},$$

where $H_g := F_g|W(Y_r)$.

Proof. Immediately,

$$\langle f_0, F_g(\cdot, s)\rangle_{\mathfrak{y}_r} = \left\langle f_0, \mathrm{Tr}^{\mathfrak{y}_r}_{\mathfrak{y}_0}(F_g)\right\rangle_{\mathfrak{y}_0}.$$

To finish we note that $W(Y_0)^2 = (-1)^{n[k]}$ and we claim

$$\mathrm{Tr}^{\mathfrak{y}_r}_{\mathfrak{y}_0}(F_g)|W(Y_0) = H_g|U_p^r,$$

the proof of which is twofold. That the matrices corresponding to the operators match up is given by the simple matrix multiplication

$$\begin{pmatrix} I_n & 0 \\ Y_0^2 u & I_n \end{pmatrix} \iota_{Y_0} = \iota_{Y_r} \begin{pmatrix} p^{-r} I_n & -p^{-r} u \\ 0 & p^r I_n \end{pmatrix},$$

for $u \in S(\mathfrak{b}^{-2}/p^{2r}\mathfrak{b}^{-2})$ and in which we used $Y_r = Y_0 p^r$. For the claim to be true however, we need to check that the factors of automorphy match up as well and, given the strong automorphy property of the traditional factor of automorphy $j(\alpha, z)$, it suffices to check the identity

$$h\left(\begin{pmatrix} I_n & 0 \\ Y_0^2 u & I_n \end{pmatrix}, \iota_{Y_0} z\right) h(\iota_{Y_0}, z) = h(\iota_{Y_r}, \alpha_u z) J^{\frac{1}{2}}(\alpha_u, z), \tag{4.3.4}$$

where

$$\alpha_u := \begin{pmatrix} p^{-r} I_n & -p^{-r} u \\ 0 & p^r I_n \end{pmatrix}.$$

By considering $\iota_M \in P_{\mathbb{A}}\iota$, using a combination of the weak automorphic property of h (see (2.1.3)), its values on $P_{\mathbb{A}}$ (see (2.1.2)), and Equation (2.5) of [Shi93, p. 1025] we see that $h(\iota_M, z) = |Miz|^{\frac{1}{2}}$. To calculate $J^{\frac{1}{2}}(\alpha_u, z)$ write $\alpha_u = \mathfrak{z}\xi$, where $\mathfrak{z} \in Z_0$ and $\xi \in D[2,2]$ are defined by $\mathfrak{z}_\infty = I_{2n}$,

$$\begin{aligned} \mathfrak{z}_q &:= \begin{pmatrix} p^{-r} I_n & 0 \\ 0 & p^r I_n \end{pmatrix}, && q \in \mathbf{f}, \\ \xi_q &:= \begin{pmatrix} I_n & -u \\ 0 & I_n \end{pmatrix} \in D_q[2,2], && q \in \mathbf{f}, \\ \xi_\infty &:= \alpha_u \in Sp_n(\mathbb{R}). \end{aligned}$$

By Definition 2.2.2 and (2.1.2) we see that $J^{\frac{1}{2}}(\alpha_u, z) = h(\xi, z) = p^{\frac{rn}{2}}$. Therefore, using $Y_r = Y_0 p^r$ and $\alpha_u z = p^{-2r}(z-u)$, the right-hand side of (4.3.4) is $|Y_0 i(z-u)|^{\frac{1}{2}}$. Finally, by use of Lemma 2.2 of [Shi93] or Proposition 2.1.2 of this thesis, we have

$$h\left(\begin{pmatrix} I_n & 0 \\ Y_0^2 u & I_n \end{pmatrix}, \iota_{Y_0} z\right) = \left|-\frac{u}{z} + I_n\right|^{\frac{1}{2}}.$$

So the left-hand side of (4.3.4) above is also $|Y_0 i(z-u)|^{\frac{1}{2}}$ and the claim, and therefore the proposition, holds. □

As an example, assume that χ is primitive, $\ell = \ell_\chi \geq 2^n - 1$, and $g = \theta_{\bar{\chi}}$. Let $H_\chi := H_{\theta_{\bar{\chi}}}$ and $V_\chi := \mathrm{Vol}(\Gamma[\mathfrak{b}^{-2}, \mathfrak{b}^2\mathfrak{y}_\chi]\backslash\mathbb{H}_n)$. Then taking $r := \ell_\chi - 2^n - 1 \geq 0$, we have $\mathfrak{y}_r = \mathfrak{y}_\chi$ and $\Gamma_r = \Gamma_\chi$, and applying the above proposition to (4.3.2), we get

$$\begin{aligned} L_\psi(s, f_0, \bar{\chi}) = {} & \left[\Gamma_n\left(\tfrac{s-n-1+k+\mu}{2}\right) 2c_{f_0}(\tau, 1)\right]^{-1} N(\mathfrak{b})^{n(n+1)} |4\pi\tau|^{\frac{s-n-1+k+\mu}{2}} \\ & \times (-1)^{n[k]} \Lambda_\tau(s) \mathcal{G}_\tau(s) \langle f_0, H_\chi | U_p^{\ell_\chi - 2^n - 1} W(Y_0)\rangle_{\mathfrak{y}_0} V_\chi. \end{aligned}$$

4.4 A transformation formula of theta series

Notation. k – half-integral weight; $[k] = k - \frac{1}{2} \in \mathbb{Z}$.
$\mathfrak{b}$ – fractional ideal of $\mathbb{Q}$; $\mathfrak{c}$ – integral ideal of $\mathbb{Z}$.
$(\mathfrak{b}^{-1}, \mathfrak{bc}) \subseteq 2\mathbb{Z} \times 2\mathbb{Z}$.
$\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}]$.
Hecke character $\psi : \mathbb{I}_\mathbb{Q}/\mathbb{Q}^\times \to \mathbb{T}$ satisfying (2.1.5) and (2.1.6).
$S_+ = \{\xi \in M_n(\mathbb{Q}) \mid \xi^T = \xi, \xi \geq 0\}$; $\tau \in S_+$.
$\mathfrak{t}$ – integral ideal such that $h^T(2\tau)^{-1}h \in 4\mathfrak{t}^{-1}$ for all $h \in \mathbb{Z}^n$.
$\hat{\tau} = N(\mathfrak{t})(2\tau)^{-1}$.
p – an odd prime.
χ – Hecke character of conductor $p^{\ell_\chi}\mathbb{Z}$.
$Y_\chi = Y_0 p^{\ell_\chi - 2^n - 1} = N(\mathfrak{tbc})p^{\ell_\chi}$.
$\iota_M = \left(\begin{smallmatrix} 0 & -M^{-1}I_n \\ MI_n & 0 \end{smallmatrix}\right)$; $g|W(M) = g||_\kappa \iota_M$.
$d = \frac{n^2}{2}$ if n is even, $d = 0$ if n is odd.

Transformation formulae for theta series of the form $\theta_\chi|W(Y_\chi)$, when χ is a primitive Dirichlet character, are generally well-known entities. The precise formula of this section is encompassed by the generality of both Theorem A3.3 and Proposition A3.17 of [Shi00]; what follows is a concrete derivation and calculation of the integrals found there.

Let $\mathcal{S}(M_n(\mathbb{Q}_\mathbf{f}))$ denote the Schwartz-Bruhat space of locally constant and compactly supported functions $M_n(\mathbb{Q}_\mathbf{f}) \to \mathbb{C}$. Explicitly, these are the functions $\lambda = \prod_p \lambda_p$ such that each $\lambda_p : M_n(\mathbb{Q}_p) \to \mathbb{C}$ is locally constant and $\lambda_p = 1_{M_n(\mathbb{Z}_p)}$ is the characteristic function of $M_n(\mathbb{Z}_p)$ for all but finitely many p. Theorem A3.3 tells us that there is a $\mathbb{C}$-linear automorphism $\lambda \to {}^\sigma\lambda$ of $\mathfrak{M}$ on $\mathcal{S}(M_n(\mathbb{Q}_\mathbf{f}))$; moreover it gives formulae for this action by $P_\mathbb{A}$ and the inversion $\iota = \left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right)$, which latter was used way back in the definition of the symplectic group.

Chapter A3 of [Shi00] concerns more general theta series of the form

$$\theta(z, \lambda) := \sum_{x \in M_n(\mathbb{Q})} \lambda(x_\mathbf{f})|x|^\mu e_\infty(\mathrm{tr}(x^T \tau x z)), \tag{4.4.1}$$

for a fixed $\tau \in S_+$, $\mu \in \{0, 1\}$ and $\lambda \in \mathcal{S}(M_n(\mathbb{Q}_\mathbf{f}))$; be apprised that putting $\lambda = \prod_p \lambda_p$ and

$$\lambda_p(y) := \begin{cases} 1 & \text{if } y \in M_n(\mathbb{Z}_p) \text{ and } p \nmid \mathfrak{f}, \\ \chi_p(|y|) & \text{if } y \in GL_n(\mathbb{Z}_p) \text{ and } p \mid \mathfrak{f}, \\ 0 & \text{otherwise,} \end{cases} \tag{4.4.2}$$

for a Hecke character χ of conductor $\mathfrak{f}$ gives the series $\theta(z, \lambda) = \theta_\chi^{(\mu)}(z; \tau)$ of (2.1.9).

Assume that $\chi : \mathbb{I}_\mathbb{Q}/\mathbb{Q}^\times \to \mathbb{T}$ is a Hecke character of conductor $p^{\ell_\chi}\mathbb{Z}$ and let $\iota_\chi = \iota_{Y_\chi}$ be the matrix of (4.3.3) with $M = Y_\chi$. Since $\iota_{Y_\chi} \in C^\theta$, Proposition A3.17, and Equation

(A3.18) in particular, of [Shi00] give that

$$\theta(z,\lambda)|W(Y_\chi) = \sum_{x\in M_n(\mathbb{Q})} \left({}^{\iota_\chi^{-1}}\lambda\right)(x_{\mathbf{f}})|x|^{\mu} e_\infty(\mathrm{tr}(x^T\tau x z)), \tag{4.4.3}$$

and so we calculate ${}^{\iota_\chi^{-1}}\lambda$. Note that

$$\iota_\chi^{-1} = \begin{pmatrix} 0 & Y_\chi^{-1}I_n \\ -Y_\chi I_n & 0 \end{pmatrix} = \iota\sigma, \qquad \sigma := \begin{pmatrix} -Y_\chi I_n & 0 \\ 0 & -Y_\chi^{-1}I_n \end{pmatrix} \in P,$$

so that by Theorem A3.3 (2) we have ${}^{\iota_\chi^{-1}}\lambda = {}^{\iota}({}^{\sigma}\lambda)$. Recall

$$d = \begin{cases} \frac{n^2}{2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

from Theorem C. Let $d_p y$ be the Haar measure on $M_n(\mathbb{Q}_p)$ such that the measure of $bM_n(\mathbb{Z}_p)$ is $|b|_p^{n^2/2}$ for any $b\in\mathbb{Q}$. The equations in Theorem A3.3 (5) and (A3.3) of [Shi00] give the first line of the following calculation

$$\begin{aligned}
{}^{\iota}({}^{\sigma}\lambda)_p(x) &= i^d |Y_\chi|_p^{\frac{n^2}{2}} |\det(2\tau)|_p^{\frac{n}{2}} \int_{M_n(\mathbb{Q}_p)} \lambda_p(-Y_\chi y) e_p(-\mathrm{tr}(x^T 2\tau y)) d_p y \\
&= i^d \chi_\infty(-1)^n |Y_\chi|_p^{\frac{n}{2}} |\det(2\tau)|_p^{\frac{n}{2}} \int_{Y_\chi^{-1}GL_n(\mathbb{Z}_p)} \chi_p(|Y_\chi y|) e_p(-\mathrm{tr}(x^T 2\tau y)) d_p y \\
&= i^d \chi_\infty(-1)^n |\det(2\tau)|_p^{\frac{n}{2}} \int_{GL_n(\mathbb{Z}_p)} \chi_p(|y|) e_p\left(-\frac{\mathrm{tr}(x^T 2\tau y)}{Y_\chi}\right) d_p y,
\end{aligned}$$

using the definition of (4.4.2) in the second line and the change of variables $y\mapsto Y_\chi y$ in the last. By the fact that $Y_\chi = N(\mathfrak{tbc})p^{\ell_\chi}$ and by the definition of $\hat{\tau}$ from (4.3.1), we have that ${}^{\iota}({}^{\sigma}\lambda)_p(x)$ is equal to

$$i^d\chi_\infty(-1)^n|\det(2\tau)|_p^{\frac{n}{2}} \sum_{a\in GL_n(\mathbb{Z}/p^{\ell_\chi}\mathbb{Z})} \chi_p(|a|) e\left(\frac{\mathrm{tr}(x^T\hat{\tau}^{-1}a)}{N(\mathfrak{bc})p^{\ell_\chi}}\right) \int_{p^{\ell_\chi}GL_n(\mathbb{Z}_p)} e_p\left(-\frac{\mathrm{tr}(x^T\hat{\tau}^{-1}y)}{N(\mathfrak{bc})p^{\ell_\chi}}\right) d_p y.$$

The integral in the above equation is non-zero if and only if the integrand is a constant function in y – i.e. if and only if $x\in |N(\mathfrak{bc})|_p^{-1}\hat{\tau}M_n(\mathbb{Z}_p)$ – at which point it is $p^{-\ell_\chi\frac{n^2}{2}}$. Likewise, for $q\nmid\mathfrak{f}$, we have ${}^{\iota}({}^{\sigma}\lambda)_q(x)\neq 0$ if and only if $x\in|N(\mathfrak{bc})|_q^{-1}\hat{\tau}M_n(\mathbb{Z}_q)$, at which point it is $|\det(2\tau)|_q^{\frac{n}{2}}$. Therefore ${}^{\iota}({}^{\sigma}\lambda)(x)\neq 0$ if and only if $x\in N(\mathfrak{bc})\hat{\tau}M_n(\mathbb{Z})$, for which

$${}^{\iota}({}^{\sigma}\lambda)(x) = i^d\chi_\infty(-1)^n|2\tau|^{-\frac{n}{2}}p^{-\ell_\chi\frac{n^2}{2}}G_n(N(\mathfrak{bc})^{-1}\hat{\tau}^{-1}x,\bar{\chi}), \tag{4.4.4}$$

where, for any Hecke character φ of conductor $\mathfrak{f}$ and $X\in M_n(\mathbb{Z})$,

$$G_n(X,\varphi) := \sum_{a\in M_n(\mathbb{Z}/N(\mathfrak{f})\mathbb{Z})} \varphi_{\mathfrak{f}}^{-1}(|a|) e\left(\frac{\mathrm{tr}(X^Ta)}{N(\mathfrak{f})}\right)$$

denotes the generalised n-degree Gauss sum. The Gauss sum of (2.1.4) relates to the above by $G_n(\varphi) = G_n(I_n,\varphi)$, and if φ is a primitive Dirichlet character then we have $G_n(X,\varphi) = \varphi^{-1}(|X|)G_n(\varphi)$ when $(|X|,N(\mathfrak{f}))=1$, and $G_n(X,\varphi)=0$ otherwise. So, under the assumption that χ is a primitive Dirichlet character and that $x\in N(\mathfrak{bc})\hat{\tau}M_n(\mathbb{Z})$, (4.4.4) above becomes

$$\iota({}^{\sigma}\lambda)(x) = i^d\chi(-1)^n|2\tau|^{-\frac{n}{2}}p^{-\ell_\chi\frac{n^2}{2}}\chi(|N(\mathfrak{bc})^{-1}\hat{\tau}^{-1}x|)G_n(\bar{\chi}).$$

Thus when χ is a primitive Dirichlet character the transformation formula of (4.4.3) translates to

$$\theta_\chi|W(Y_\chi) = \frac{i^d\chi(-1)^n}{|2\tau|^{\frac{n}{2}}}p^{-\ell_\chi\frac{n^2}{2}}G_n(\bar{\chi})\sum_{x\in N(\mathfrak{bc})\hat{\tau}M_n(\mathbb{Z})}\chi(|N(\mathfrak{bc})^{-1}\hat{\tau}^{-1}x|)|x|^\mu e_\infty(\mathrm{tr}(x^T\tau xz)),$$

and this becomes, by writing $x = N(\mathfrak{bc})\hat{\tau}x'$, the desired formula:

$$\theta_\chi^{(\mu)}(z;\tau)|W(Y_\chi) = \chi(-1)^n\frac{i^dN(\mathfrak{tbc})^{n\mu}}{|2\tau|^{\frac{n}{2}+\mu}}p^{-\ell_\chi\frac{n^2}{2}}G_n(\bar{\chi})\theta_{\bar{\chi}}^{(\mu)}\left(N(\sqrt{\mathfrak{t}\mathfrak{bc}})^2\frac{z}{2};\hat{\tau}\right), \tag{4.4.5}$$

where we have written $N(\sqrt{\mathfrak{a}}) = |N(\mathfrak{a})^{\frac{1}{2}}|$ for any integral ideal $\mathfrak{a}$.

4.5 Fourier expansions of Eisenstein series

Notation. k – half-integral weight; $[k] = k - \frac{1}{2} \in \mathbb{Z}$.
$\mathfrak{b}$ – fractional ideal of $\mathbb{Q}$; $\mathfrak{c}$ – integral ideal of $\mathbb{Z}$.
$(\mathfrak{b}^{-1}, \mathfrak{bc}) \subseteq 2\mathbb{Z}\times 2\mathbb{Z}$.
$\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}]$.
Hecke character $\psi : \mathbb{I}_\mathbb{Q}/\mathbb{Q}^\times \to \mathbb{T}$ satisfying (2.1.5) and (2.1.6).
$\delta = n \pmod 2 \in \{0, 1\}$.
S^∇ – set of all symmetric half-integral $n\times n$ matrices.
$S_+ = \{\xi \in M_n(\mathbb{Q}) \mid \xi^T = \xi, \xi \geq 0\}$; $\tau \in S_+$.
$\mathfrak{t}$ – integral ideal such that $h^T(2\tau)^{-1}h \in 4\mathfrak{t}^{-1}$ for all $h \in \mathbb{Z}^n$.
$\hat{\tau} = N(\mathfrak{t})(2\tau)^{-1}$.
p – an odd prime.
χ – Hecke character of conductor $p^{\ell_\chi}\mathbb{Z}$.
$0 \leq r \in \mathbb{Z}$; $\mathfrak{y}_r = \mathfrak{y}_0p^{2r} = [\mathfrak{tc}p^{2^n-1+r}]^2$; $Y_r = N(\mathfrak{b})\sqrt{N(\mathfrak{y}_r)}$; $\Gamma_r = \Gamma[\mathfrak{b}^{-2}, \mathfrak{b}^2\mathfrak{y}_r]$.
$\iota_M = \left(\begin{smallmatrix} 0 & -M^{-1}I_n \\ MI_n & 0\end{smallmatrix}\right)$; $g|W(M) = g||_\kappa\iota_M$.
$N(\sqrt{\mathfrak{a}}) = |N(\mathfrak{a})^{\frac{1}{2}}|$ for an integral ideal $\mathfrak{a}$.
$X_p = \{x \in \mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times | x \text{ is continuous}\}$.
$\mathbf{Pr}$ – holomorphic projection map of Theorem 3.2.2.

Given Proposition 4.3.1 and the transformation formula in (4.4.5), it will be germane to give explicit expressions for the Fourier coefficients of $\mathbf{Pr}(H^\star|U_p^r)$, where $r \geq 0$, $H^\star$ is the form

$$H^\star(z, s) = \theta_\chi^\star(z)\mathcal{E}_{\bar{\eta}}^\star(z, \tfrac{2s-n}{4}),$$

and

$$\theta_\chi^\star(z) := \theta_\chi^{(\mu)}\left(N(\sqrt{\mathfrak{t}\mathfrak{bc}})^2\frac{z}{2};\hat{\tau}\right), \tag{4.5.1}$$

$$\mathcal{E}^{\star}_{\bar\eta}(z,s) := \mathcal{E}_{k-\frac{n}{2}-\mu}(z,s;\bar\eta,\Gamma_r)|W(Y_r). \tag{4.5.2}$$

Recall the notation $\eta = \psi\bar\chi\rho_\tau$ and $\delta = n \pmod 2 \in \{0,1\}$.

Proposition 4.5.1. *For any $\varsigma \in S_+$ define*

$$V_\varsigma := \left\{(\varsigma_1,\varsigma_2) \in M_n(\mathbb{Z}) \times S_+ \middle| \frac{N(\sqrt{\mathfrak{t}}\mathfrak{b}\mathfrak{c})^2}{2}\varsigma_1^T\hat\tau\varsigma_1 + \varsigma_2 = \varsigma\right\}.$$

Assume that $k > 2n$, χ is a Dirichlet character, and $m \in \Omega_{n,k}$. For any $\beta \in \mathbb{Z}$, there exists a polynomial $P(\sigma,\sigma';\beta) \in \mathbb{Q}[\sigma_{ij},\sigma'_{ij} \mid 1 \le i,j \le n]$, defined for $\sigma,\sigma' \in S_+$; a finite subset $\mathbf{c}$ of primes; polynomials $f_{\sigma,q} \in \mathbb{Z}[t]$, defined for each $\sigma \in S_+$ and $q \in \mathbf{c}$, whose coefficients are independent of χ; and a factor

$$C^{\star}_{\pm}(\sigma,m) := i^{-n([k-\frac{n}{2}-\mu])}N(\mathfrak{b}^2\mathfrak{y}_r)^{n(\frac{3n-2m}{4}-k+\mu)}2^{n(k-\mu+\frac{3}{2})}\pi^{m(\frac{m+k-n-\mu}{2})}$$
$$\times\,\Gamma_n(\tfrac{m+k-n-\mu}{2})^{-1}|\sigma|^{m_\pm}\prod_{q\in\mathbf{c}} f_{\sigma,q}(\bar\eta(q)q^{\frac{n+\delta-1}{2}-m}),$$

where $m_+ = m-n-\frac{1}{2}$, $m_- = 0$, such that if $m \in \Omega_{n,k}\backslash\{n+\frac{1}{2}\}$ (and $m \ne n+\frac{3}{2}$ if $n>1$ and $(\psi^*\chi)^2 = 1$), then $\mathbf{Pr}(\theta^{\star}_{\chi}\mathcal{E}^{\star}_{\bar\eta}(\cdot,\frac{2m-n}{4})|U_p^r)$ has non-zero Fourier coefficients only when $\sigma > 0$, for which we have

$$c\left(\sigma,1;\mathbf{Pr}(\theta^{\star}_{\chi}\mathcal{E}^{\star}_{\bar\eta}(\cdot,\tfrac{2m-n}{4})|U_p^r)\right) = \sum_{(\sigma_1,\sigma_2)\in V_{p^r\sigma}} \chi(|\sigma_1|)|\sigma_1|^{\mu}C^{\star}_{+}(\sigma_2,m)P(\sigma_2,p^r\sigma;\tfrac{k-m-\mu}{2})$$

if $m \in \Omega^+_{n,k}$, whereas

$$c\left(\sigma,1;\mathbf{Pr}(\theta^{\star}_{\chi}\mathcal{E}^{\star}_{\bar\eta}(\cdot,\tfrac{2m-n}{4})|U_p^r)\right) = \sum_{(\sigma_1,\sigma_2)\in V_{p^r\sigma}} \chi(|\sigma_1|)|\sigma_1|^{\mu}C^{\star}_{-}(\sigma_2,m)P(\sigma_2,p^r\sigma;\tfrac{k+m-\mu-1-2n}{2})$$

if $m \in \Omega^-_{n,k}$.

Furthermore, the polynomial $P(\sigma,\sigma';\beta)$ satisfies $P(\sigma,\sigma';\beta) \equiv |\sigma|^\beta \pmod{\sigma'_{ij}}$ for $\beta \in \mathbb{Z}$.

When k is an integer and n is even the above kind of result is well known, see for example Theorem 4.6 of [Pan91, p. 77]. Since we have seen, in Theorem 3.2.2, that the definition of the projection map remains the same for half-integral weights, we can obtain the above in a similar manner, by using results on the Fourier development of integral and half-integral weight Eisenstein series as follows.

Suppose that $\kappa \in \frac{1}{2}\mathbb{Z}$ is such that $2\kappa + n \notin 2\mathbb{Z}$ and recall the normalisation $\mathcal{E}_\kappa(z,s)$ of (2.4.2), taking the congruence subgroup $\Gamma_0 = \Gamma[\mathfrak{x}^{-1},\mathfrak{x}\mathfrak{y}]$. Further assume that $N(\mathfrak{x})$, $N(\mathfrak{y})$ are both squares, that $(\mathfrak{x}^{-1},\mathfrak{x}\mathfrak{y}) \subseteq 2\mathbb{Z}\times 2\mathbb{Z}$ if $\kappa \notin \mathbb{Z}$, and let $Y := \sqrt{N(\mathfrak{x}\mathfrak{y})}$. If $s \in \frac{1}{4}\mathbb{Z}$, $s \ne \frac{n+1}{4}$, and $s \ne \frac{n+3}{4}$ if $n > 1$ and $\eta^2 = 1$, then Proposition 17.6 of [Shi00] combined with both the analytic continuation of the Eisenstein series and the fact that $\iota\Gamma_0\iota = \Gamma[\mathfrak{x}\mathfrak{y},\mathfrak{x}^{-1}]$ gives

$$\mathcal{E}_\kappa(z,s)|\iota = \sum_{\substack{0<\sigma\in S_+\\ N(\mathfrak{x}\mathfrak{y})\sigma\in S^\triangledown}} c^{\pm}(\sigma,y,s)e_\infty(\sigma x),$$

where, if $\sigma > 0$, we have by Propositions 16.9 and 16.10 of [Shi00] that

$$c^{\pm}(\sigma, y, s) = i^{n(\kappa-[\kappa])} N(\mathfrak{x}\mathfrak{y})^{-\frac{n(n+1)}{2}} |y|^{s-\frac{\kappa}{2}} \xi(y, \sigma; s + \tfrac{\kappa}{2}, s - \tfrac{\kappa}{2}) \prod_{q \in \mathbf{c}} f_{Y^2\sigma, q}(\bar{\eta}(q) q^{-2s+[\kappa]-\kappa}),$$

$$\xi(g, h; s, s') := \int_{S_\infty} e_\infty(-hx) |x - ig|^{-s} |x - ig|^{-s'} dx,$$

for $g \in Y$ (which set is defined in (2.1.12)), $h \in S_\infty$, and $s, s' \in \mathbb{C}$. The aim now is to represent the function ξ at certain values as a polynomial to which one can easily apply the definition of holomorphic projection. This is done via several intermediary relations and functions, the first of which relates ξ to the hyperconfluent geometric function $\omega(g, h; s, s')$ of [Shi82] and is given (the relation) in (17.11) of [Shi00] as

$$\begin{aligned} \xi(y, \sigma; s + \tfrac{\kappa}{2}, s - \tfrac{\kappa}{2}) =& i^{-n\kappa} 2^{n(\kappa+1)} \pi^{n(s+\frac{\kappa}{2})} \Gamma_n(s + \tfrac{\kappa}{2})^{-1} |y|^{\frac{\kappa}{2}-s} |\sigma|^{s+\frac{\kappa-n-1}{2}} \\ & \times \omega(2\pi y, \sigma; s + \tfrac{\kappa}{2}, s - \tfrac{\kappa}{2}). \end{aligned} \tag{4.5.3}$$

Then the function $\omega(g, h; s, s')$ is continued analytically by defining

$$\begin{aligned} \zeta(g; s, s') &:= \int_Y e^{-\operatorname{tr}(gx)} |x + I_n|^{s-\kappa} |x|^{s'-\kappa} dx, \\ \omega_0(g; s, s') &:= \Gamma_n(s')^{-1} |g|^{s'} \zeta(g; s, s'), \end{aligned}$$

which latter function has an analytic continuation, see Theorem 3.1 of [Shi82]. By certain properties of these functions, see (4.7.K) of [Shi82] for the first line, and [Shi82, (4.10)] plus the definition of ω_0 for the second line, one has

$$\begin{aligned} \omega(2\pi y, \sigma; s + \tfrac{\kappa}{2}, s - \tfrac{\kappa}{2}) &= \omega(4\pi \tilde{a} y a^{-1}, I_n; s + \tfrac{\kappa}{2}, s - \tfrac{\kappa}{2}) \\ &= 2^{\frac{n(n+1)}{2}} e^{-2\pi \operatorname{tr}(\sigma y)} \omega_0(4\pi\sigma y; s + \tfrac{\kappa}{2}, s - \tfrac{\kappa}{2}), \end{aligned} \tag{4.5.4}$$

where $a \in GL_n(\mathbb{R})$ satisfies $a\sigma a^T = I_n$ (and therefore $\sigma = a^{-1}\tilde{a}$). It is this function ω_0 that has a polynomial representation in terms of the differential operator

$$\begin{aligned} R(g; \beta, s') &:= (-1)^{n\beta} e^{\operatorname{tr}(g)} |z|^{\beta+s'} \det \left[\frac{\partial_n}{\partial_n g}\right]^\beta (e^{-\operatorname{tr}(g)} |z|^{-s'}), \\ \frac{\partial_n}{\partial_n g} &:= \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial g_{ij}}\right)_{i,j=1}^n, \end{aligned}$$

for $s' \in \mathbb{C}$ and $0 \le \beta \in \mathbb{Z}$. Now Proposition 3.2 of [Shi82] and the proposition of [Pan91, p. 64] tell us that, if $0 \le \frac{\kappa}{2} - s \in \mathbb{Z}$, then

$$\omega_0(4\pi\sigma y; s + \tfrac{\kappa}{2}, s - \tfrac{\kappa}{2}) = |4\pi\sigma y|^{s-\frac{\kappa}{2}} R(4\pi\sigma y; \tfrac{\kappa}{2} - s, \tfrac{n+1-\kappa}{2} - s) \tag{4.5.5}$$

whereas, if $0 \le s + \frac{\kappa-n-1}{2} \in \mathbb{Z}$, then

$$\omega_0(4\pi\sigma y; s + \tfrac{\kappa-n-1}{2} + \tfrac{n+1}{2}; s - \tfrac{\kappa}{2}) = |4\pi\sigma y|^{-s-\frac{\kappa-n-1}{2}} R(4\pi\sigma y; s + \tfrac{\kappa-n-1}{2}, s - \tfrac{\kappa}{2}). \tag{4.5.6}$$

Combining both (4.5.3) and (4.5.4) with (4.5.5) (resp. (4.5.6)) in the case $0 \le \frac{\kappa}{2} - s \in \mathbb{Z}$

and $s > \frac{n+1}{4}$ (resp. $0 \le s + \frac{\kappa-n-1}{2} \in \mathbb{Z}$ and $s < \frac{n+1}{4}$) therefore gives

$$\begin{aligned} c^+(\sigma, y, s) &= C(\sigma, s)|4\pi\sigma y|^{s-\frac{\kappa}{2}} R(4\pi\sigma y; \tfrac{\kappa}{2} - s, \tfrac{n+1-\kappa}{2} - s)e^{-2\pi\operatorname{tr}(\sigma y)}, \\ c^-(\sigma, y, s) &= C(\sigma, s)|4\pi\sigma y|^{\frac{n+1-\kappa}{2}-s} R(4\pi\sigma y; s + \tfrac{\kappa-n-1}{2}, s - \tfrac{\kappa}{2})e^{-2\pi\operatorname{tr}(\sigma y)}, \\ C(\sigma, s) &: = i^{-n[\kappa]} N(\mathfrak{x}\mathfrak{y})^{-\frac{n(n+1)}{2}} 2^{n(\kappa+\frac{n+3}{2})} \pi^{n(s+\frac{\kappa}{2})} \Gamma_n(s + \tfrac{\kappa}{2})^{-1} \\ &\quad \times |\sigma|^{s+\frac{\kappa-n-1}{2}} \prod_{q\in\mathbf{c}} f_{Y^2\sigma,q}(\bar{\eta}(q)q^{-2s+[\kappa]-\kappa}). \end{aligned} \tag{4.5.7}$$

Now since $\mathcal{E}_\kappa(z, s)|W(Y) = Y^{-n\kappa}\mathcal{E}_\kappa(z, s)|\iota\big|_{z=Y^2z}$ we have that

$$\mathcal{E}_\kappa(z, s)|W(Y) = \sum_{0<\sigma\in S^\triangledown} c^\pm_\mathfrak{y}(\sigma, y, s)e_\infty(\sigma x),$$

where $c^\pm_\mathfrak{y}(\sigma, y, s) := Y^{-n\kappa}c^\pm(Y^{-2}\sigma, Y^2y, s)$ are given explicitly by

$$\begin{aligned} c^+_\mathfrak{y}(\sigma, y, s) &= Y^{-n\kappa}C(Y^{-2}\sigma, s)|4\pi\sigma y|^{s-\frac{\kappa}{2}} R(4\pi\sigma y; \tfrac{\kappa}{2} - s, \tfrac{n+1-\kappa}{2} - s)e^{-2\pi\operatorname{tr}(\sigma y)}, \\ c^-_\mathfrak{y}(\sigma, y, s) &= Y^{-n\kappa}C(Y^{-2}\sigma, s)|4\pi\sigma y|^{\frac{n+1-\kappa}{2}-s} R(4\pi\sigma y; s - \tfrac{n+1-\kappa}{2}, s - \tfrac{\kappa}{2})e^{-2\pi\operatorname{tr}(\sigma y)}. \end{aligned}$$

Now put $C^+_\mathfrak{y}(\sigma, s) := Y^{-n\kappa}|\sigma|^{s-\frac{\kappa}{2}}C(Y^{-2}\sigma, s)$ and $C^-_\mathfrak{y}(\sigma, s) := Y^{-n\kappa}|\sigma|^{\frac{n+1-\kappa}{2}-s}C(Y^{-2}\sigma, s)$. By analogy with the proof of Theorem 4.6 of [Pan91, p. 77], the application of the holomorphic projection combined with (4.5.7) gives that if $g \in \mathcal{M}_\ell$ is a holomorphic modular form, then

$$c(\sigma, 1; \mathbf{Pr}([g\mathcal{E}_\kappa(\cdot, s)|W(Y)]|U_p^r)) = \sum_{\sigma_1,\sigma_2} c_g(\sigma_1, 1)C^+_\mathfrak{y}(\sigma, s)P\left(\sigma_2, p^r\sigma; \tfrac{\kappa}{2} - s\right) \tag{4.5.8}$$

when $0 \le \frac{\kappa}{2} - s \in \mathbb{Z}$, $s > \frac{n+1}{4}$ (and $s \ne \frac{n+3}{4}$ if $\eta^2 = 1$ and $n > 1$), and $\sigma > 0$, whereas

$$c(\sigma, 1; \mathbf{Pr}([g\mathcal{E}_\kappa(\cdot, s)|W(Y)]|U_p^r)) = \sum_{\sigma_1,\sigma_2} c_g(\sigma_1, 1)C^-_\mathfrak{y}(\sigma, s)P(\sigma_2, p^r\sigma; s + \tfrac{\kappa-n-1}{2}) \tag{4.5.9}$$

when $0 \le s + \frac{\kappa-n-1}{2} \in \mathbb{Z}$, $s < \frac{n+1}{4}$, and $\sigma > 0$; in both cases the summation is taken over all $\sigma_1, \sigma_2 \in S_+$ such that $\sigma_1 + \sigma_2 = p^r\sigma$.

Specialising (4.5.8) and (4.5.9) to the case $\mathfrak{x} = \mathfrak{b}^2, \mathfrak{y} = \mathfrak{y}_r, \kappa = k - \frac{n}{2} - \mu, \ell = \frac{n}{2} + \mu, g = \theta^\star_\chi$, and $s = \frac{2m-n}{4}$ for $m \in \Omega^\pm_{n,k}$ gives Proposition 4.5.1 by putting $C^\star_\pm(\sigma, m) := C^\pm_{\mathfrak{y}_r}(\sigma, \frac{2m-n}{4})$.

Proposition 4.5.2. *There exist p-adic distributions $\Sigma_{\sigma,m}$ defined, for any $m \in \frac{1}{2}\mathbb{Z}$ such that $m - \frac{1}{2} \in \mathbb{Z}$, by*

$$\Sigma_{\sigma,m}(\chi) := \iota_p \left[\prod_{p\ne q\in\mathbf{c}} f_{\sigma,q}(\bar{\chi}(q)q^{\frac{n+\delta-1}{2}-m}) \right].$$

If we set $\Sigma_\sigma = \Sigma_{\sigma,\frac{1}{2}}$ then this defines a p-adic measure that satisfies

$$\int_{\mathbb{Z}_p^\times} \chi x_p^{[m]} d\Sigma_\sigma = \Sigma_{\sigma,m}(\chi),$$

for any $m \in \frac{1}{2}\mathbb{Z}$ such that $[m] = m - \frac{1}{2} \in \mathbb{Z}$.

Proof. That $\Sigma_{\sigma,m}$ satisfies the compatibility criterion, and therefore defines a $\overline{\mathbb{Q}}$-valued distribution, is evident. That Σ_σ is p-adically bounded is also obvious, since it is a finite product of polynomials whose coefficients are independent of χ, and this proves the proposition. It can also be proved using the Kummer congruences, which proof, to pre-empt the use of the congruences later on, is now detailed.

Take $\chi x_p^{[m]}$ for $\chi \in X_p^{\mathrm{tors}}$ and $m \in \frac{1}{2}\mathbb{Z}$ with $m - \frac{1}{2} \in \mathbb{Z}$ as the system $\{f_i\}$ in the statement of the Kummer congruences. Taking $\mathfrak{Re}(s) \to \infty$ in the identity (16.46) of [Shi00] tells us that $\prod_{p \neq q \in \mathbf{c}} f_{\sigma,q}(0) = 0$ and so this product of polynomials has no constant term. We may then write

$$\prod_{p \neq q \in \mathbf{c}} f_{\sigma,q}(\bar{\chi}(q) q^{\frac{n+\delta-1}{2} - m}) = \sum_{j=1}^{e} a_j c_j^{j(\frac{n+\delta-2}{2})} (\chi x_p^{[m]})(c_j^{-j}),$$

where $0 < e \in \mathbb{Z}$, $a_j \in \mathbb{Z}$ are independent of χ and $c_j \in \mathbb{Z}$ is some product of primes $q \neq p$ in $\mathbf{c}$. Hence given any finite subset $\mathcal{X} \subseteq X_p^{\mathrm{tors}}$ and the assumption, for some $\{b_\chi\} \subseteq \mathcal{O}_p$, that

$$\sum_{\chi \in \mathcal{X}} b_\chi \chi x_p^{[m]} \subseteq p^N \mathcal{O}_p,$$

all the while suppressing the notation of ι_p, one has

$$\begin{aligned} \sum_{\chi \in \mathcal{X}} b_\chi \int_{\mathbb{Z}_p^\times} \chi x_p^{[m]} d\Sigma_\sigma &= \sum_{\chi \in \mathcal{X}} b_\chi \prod_{p \neq q \in \mathbf{c}} f_{\sigma,q}(\bar{\chi}(q) q^{\frac{n+\delta-1}{2} - m}) \\ &= \sum_{j=1}^{e} a_j c_j^{j(\frac{n+\delta-2}{2})} \sum_{\chi \in \mathcal{X}} b_\chi (\chi x_p^{[m]})(c_j^{-j}) \in p^N \mathcal{O}_p, \end{aligned}$$

where we used the fact that each a_j is independent of χ, as well as the main assumption. □

4.6 p-adic interpolation

Notation. k – half-integral weight; $[k] = k - \frac{1}{2} \in \mathbb{Z}$.
$\mathfrak{b}$ – fractional ideal of $\mathbb{Q}$; $\mathfrak{c}$ – integral ideal of $\mathbb{Z}$.
$(\mathfrak{b}^{-1}, \mathfrak{bc}) \subseteq 2\mathbb{Z} \times 2\mathbb{Z}$.
$\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}]$.
Hecke character $\psi : \mathbb{I}_{\mathbb{Q}}/\mathbb{Q}^\times \to \mathbb{T}$ satisfying (2.1.5) and (2.1.6).
$S_+ = \{\xi \in M_n(\mathbb{Q}) \mid \xi^T = \xi, \xi \geq 0\}$; $\tau \in S_+$.
ρ_τ – quadratic character associated to $\mathbb{Q}(i^{[n/2]}\sqrt{|2\tau|})/\mathbb{Q}$.
$\mathfrak{t}$ – integral ideal such that $h^T(2\tau)^{-1}h \in 4\mathfrak{t}^{-1}$ for $\tau \in M_n(\mathbb{Q})$.
$\hat{\tau} = N(\mathfrak{t})(2\tau)^{-1}$.
$\Lambda_\tau(s) = (\Lambda_{\mathfrak{c}}/\Lambda_{\mathfrak{tc}})(\frac{2s-n}{4})$.
$\mathcal{G}_\tau = \prod_{q \in \mathbf{b}} g_q((\psi^{\mathfrak{c}p}\bar{\chi})(q) q^{-s})^{-1}$.
Prime number $p \nmid \mathfrak{c}$.
p-ordinary eigenform $f \in \mathcal{S}_k(\Gamma, \psi)$ – means that $|p^{\frac{n(n+1)}{2}} \lambda_{p,1} \cdots \lambda_{p,n}|_p = 1$.

Notation. f_0 – p-stabilisation of f of Definition 4.2.5.
$d = \frac{n^2}{2}$ if n is even, $d = 0$ if n is odd.
$N(\sqrt{\mathfrak{a}}) = |N(\mathfrak{a})^{\frac{1}{2}}|$ for an integral ideal $\mathfrak{a}$.
$\theta^\star_\chi(z) = \theta_\chi(N(\sqrt{\mathfrak{tbc}})^2 z/2; \hat{\tau})$.
$V_\varsigma = \{(\varsigma_1, \varsigma_2) \in M_n(\mathbb{Z}) \times S_+ \mid \frac{N(\sqrt{\mathfrak{tbc}})^2}{2} \varsigma_1^T \hat{\tau} \varsigma_1 + \varsigma_2 = \varsigma\}$, for $\varsigma \in S_+$.

This section finishes the proof of Theorem C, the main result of this chapter, and its structure is as follows. First the definition of a complex-valued distribution ν_s^+ interpolating $L_\psi(s, f, \bar{\chi})$ is given, and this is subsequently normalised so that it becomes $\overline{\mathbb{Q}}$-valued, therefore defining a p-adic distribution. This process is repeated for the other distribution ν_s^-. That they are each bounded, and so define p-adic measures, is then done in tandem, and this follows by the measure of the Fourier coefficients of the Eisenstein series – see Proposition 4.5.2.

Assume $k > 2n$ and take a p-ordinary eigenform $f \in \mathcal{S}_k(\Gamma, \psi)$. Recall the assumption that $p \nmid \mathfrak{c}$, and now assume that there exists $\tau \in S_+$ such that $c_f(\tau, 1) \neq 0$ and $c_{f_0}(\tau, 1) \neq 0$.

Proposition 4.6.1. *There exists a complex distribution ν_s^+ on $\mathbb{Z}_p^\times$, which is uniquely determined on Dirichlet characters χ of p-power conductor p^{ℓ_χ} as follows. If χ is primitive, $1 \leq \ell_\chi \in \mathbb{Z}$ and $\mu \in \{0, 1\}$ is such that $(\psi\chi)_\infty(-1) = (-1)^{[k]+\mu}$, then it is defined by*

$$\begin{aligned}\nu_s^+(\chi) := {} & \frac{(-1)^{n[k]} |2\tau|^{\frac{n}{2}+\mu}}{i^d N(\mathfrak{tbc})^{n\mu}} \left| -\frac{N(\sqrt{\mathfrak{tbc}})^2}{2} \hat{\tau} \right|^{-\frac{k+s-\mu-1-2n}{2}} \frac{p^{n\ell_\chi(n+1-k-s)} G_n(\bar{\chi})}{\Lambda_\tau(s) \mathcal{G}_\tau(s)} \\ & \times \left(p^{\frac{n(n+1)}{2}} \lambda_{p,1} \cdots \lambda_{p,n} \right)^{-\ell_\chi} L_\psi(s, f, \bar{\chi}),\end{aligned} \tag{4.6.1}$$

where the number d, $\Lambda_\tau(s) = \left(\frac{\Lambda_{\mathfrak{c}_0}}{\Lambda_{\mathfrak{y}_0}}\right)\left(\frac{2s-n}{4}\right)$ and $\mathcal{G}_\tau(s) = \prod_{q \in \mathbf{b}} g_q\left((\psi^{\mathfrak{c}_0}\bar{\chi})(q) q^{-s}\right)$ are all as in Theorem C.

In general, for any $\ell > \ell_\chi$, let χ_ℓ denote the character modulo p^ℓ associated to χ and, for any $r > \ell$, define

$$\begin{aligned}\nu_s^+(\chi) := {} & \frac{|\tau|^{\frac{s-n-1+k+\mu}{2}}}{c_{f_0}(\tau, 1)} \left| -\frac{N(\sqrt{\mathfrak{tbc}})^2}{2} \hat{\tau} \right|^{-\frac{k+s-\mu-1-2n}{2}} \Lambda_{\mathfrak{y}_0}\left(\tfrac{2s-n}{4}\right) p^{rn(\frac{3n}{2}+1-k-s)} \\ & \times \left(p^{\frac{n(n+1)}{2}} \lambda_{p,1} \cdots \lambda_{p,n} \right)^{-r} D\left(\tfrac{2s-3n-2}{4}, f_0, \theta^\star_{\chi_\ell} | W(Y_r)\right),\end{aligned} \tag{4.6.2}$$

where, recall, $D(s, f, g)$ is defined in Definition 2.4.1 and $\theta^\star_{\chi_\ell}(z)$ is defined in (4.5.1).

Proof. By Proposition 4.1.2 it is enough to show that the definition of ν_s given is independent of ℓ and r. When χ is primitive, this is immediate from the definition. When χ is imprimitive, the expression (4.6.2) is independent of ℓ since $\ell > \ell_\chi$. By definition

$$D\left(\tfrac{2s-3n-2}{4}, f_0, \theta^\star_{\chi_\ell} | W(Y_r)\right) = \sum_{\sigma \in S_+/GL_n(\mathbb{Z})} \nu_\sigma^{-1} c_{f_0}(\sigma, 1) c_{\theta^\star_{\chi_\ell}|W(Y_r)}(\sigma, 1)^\rho |\sigma|^{-\frac{s-n-1+k+\mu}{2}},$$

where $\rho \in \mathrm{Aut}(\mathbb{C})$ denotes complex conjugation. Let $V(M)$ be the operator associated to

$\left(\begin{smallmatrix} MI_n & 0 \\ 0 & M^{-1}I_n \end{smallmatrix}\right)$, which acts as $f|V(M) = M^{n[k]}|M^{\frac{n}{2}}|f(M^2z)$, and notice that

$$\theta^\star_{\chi_\ell}|W(Y_r) = \theta^\star_{\chi_\ell}|W(Y_\ell)V(p^{r-\ell})$$

therefore has coefficients

$$c(p^{2r-2\ell}\sigma, 1; \theta^\star_{\chi_\ell}|W(Y_r)) = [p^{n(\ell-r)(\frac{n}{2}+\mu)}]c_{\theta^\star_{\chi_\ell}|W(Y_\ell)}(\sigma, 1).$$

So the Dirichlet series $D(\frac{2s-3n-2}{4}, f_0, \theta^\star_{\chi_\ell}|W(Y_r))$ becomes

$$p^{n(\ell-r)(\frac{n}{2}+\mu)} \sum_{\sigma\in S_+/GL_n(\mathbb{Z})} \nu_\sigma^{-1} c_{f_0}(p^{2r-2\ell}\sigma, 1)c_{\theta^\star_{\chi_\ell}|W(Y_\ell)}(\sigma, 1)^\rho |p^{2(\ell-r)}\sigma|^{-\frac{s-n-1+k+\mu}{2}},$$

and thus the powers of p^r in the definition of the measure in (4.6.2) cancel. Now U_p shifts coefficients, which with Proposition 4.2.7 gives

$$c_{f_0}(p^{2r-2\ell}\sigma, 1) = \left(p^{\frac{n(n+1)}{2}}\lambda_{p,1}\cdots\lambda_{p,n}\right)^{r-\ell} c_{f_0}(\sigma, 1).$$

This cancels the remaining dependence on r in (4.6.2). □

Remark 4.6.2. Through the use of the following: the manipulations on $D(s, f, g)$ in the above proof; the identities in (2.4.16) and (2.4.20) relating $D(s, f_0, \theta_\chi)$ to $L_\psi(s, f_0, \chi)$; Corollary 4.2.8; the transformation formula of (4.4.5); and that $G_n(\chi)^{-1} = \chi(-1)^n p^{-n^2\ell_\chi} G(\bar{\chi})$, one can check that the two expressions indeed coincide if χ is primitive and $\ell = \ell_\chi$. The reason for including the general definition of the distribution for imprimitive characters is that, in order to use Proposition 4.1.5, we need to check the Kummer congruences for *all* characters.

The above distribution can be normalised so that it gives values in $\overline{\mathbb{Q}}$ and it therefore defines, after application of ι_p, a p-adic distribution.

Proposition 4.6.3. *Let $m \in \Omega^+_{n,k}$ and let c_m be defined by (3.1.3). If $k > 2n$ then, for any Dirichlet character χ of p-power modulus, we have*

$$\frac{\nu^+_m(\chi)}{\pi^{c_m}\langle f, f\rangle} \in \overline{\mathbb{Q}}.$$

Proof. Whenever χ is primitive and $n > 2$ this follows immediately from Theorem B2 upon the observation that, by Theorem 3.5.2, we have $\langle f, f\rangle \in \mu(\Lambda, k, \psi)\overline{\mathbb{Q}}$ and the fact that $\omega_\delta(m, \bar{\eta}) \in \overline{\mathbb{Q}}$. For arbitrary n, one can also use Theorem 28.8 from [Shi00] since we are concerned only with these values being in $\overline{\mathbb{Q}}$.

If χ is not primitive, then use the unfolded integral expression of $D(s, f, g)$ found in (2.4.10) to obtain the integral expression

$$\begin{aligned} \nu^+_m(\chi) = &\left[2c_{f_0}(\tau, 1)\Gamma_n\left(\tfrac{m-n-1+k+\mu}{2}\right)\right]^{-1} N(\mathfrak{b})^{n(n+1)} \left|-\tfrac{N(\sqrt{\mathfrak{t}}\mathfrak{b}\mathfrak{c})^2}{2}\hat{\tau}\right|^{-\frac{k+m-\mu-1-2n}{2}} \\ &\times |4\pi\tau|^{\frac{m-n-1+k+\mu}{2}} p^{rn(\frac{3n}{2}+1-k-m)}\left(p^{\frac{n(n+1)}{2}}\lambda_{p,1}\cdots\lambda_{p,n}\right)^{-r} \\ &\times \left\langle f_0, \theta^\star_{\chi_\ell}|W(Y_r)\mathcal{E}_{k-\frac{n}{2}-\mu}(\cdot, \tfrac{2m-n}{4}; \bar{\eta}, \Gamma_r)\right\rangle_{\mathfrak{y}_r} V_r. \end{aligned} \tag{4.6.3}$$

The application of Proposition 3.4.3 with $g = \theta^\star_{\chi\ell}|W(Y_r)$, analogous subsequent modifications (as in (3.4.3) and (3.4.4)) to the above expression, and the use of Theorem 3.5.2 again, just as was done in the previous chapter, proves the proposition in this case too. $\square$

Therefore the distribution ν^{0+}_m, defined below for all $m \in \frac{1}{2}\mathbb{Z}$ with $n < m \leq k - \mu$, is a p-adic distribution. If $m \in \Omega^+_{n,k}\backslash\{n+\frac{1}{2}\}$ (i.e. if $(\psi\chi)_\infty(-1) = (-1)^{[m]}$) then set

$$\nu^{0+}_m(\chi) = \iota_p\left[\frac{\nu^+_m(\chi)}{\pi^{c_m}\langle f, f\rangle}\right],$$

otherwise $\nu^{0+}_m(\chi) = 0$ (and moreover set $\nu^{0+}_{n+\frac{3}{2}}(\chi) = 0$ if $n > 1$ and $(\psi^*\chi)^2 = 1$).

The proof that ν^{0+}_m actually represents a p-adic measure invokes the Rankin-Selberg integral expression. To allow more concise expressions, we collect superfluous terms into C_r by defining

$$\begin{aligned} C_r :&= (-1)^{n\left([k-\frac{n}{2}-\mu]\right)}\left[\pi^{c_m}2c_{f_0}(\tau,1)\Gamma_n\left(\tfrac{m+k+\mu-n-1}{2}\right)\right]^{-1}N(\mathfrak{b})^{n(n+1)} \\ &\times |4\pi\tau|^{\frac{m+k+\mu-n-1}{2}}p^{rn(\frac{3n}{2}+1-k-m)}V_r. \end{aligned} \tag{4.6.4}$$

Note that this is independent of χ and the factor of (-1) appears as a result of $\theta^\star_\chi|W(Y_r)^2$ in the following calculation. Combining the integral expression in (4.6.3) above with Proposition 4.3.1 for $g = \theta^\star_{\chi\ell}|W(Y_r)$ and $r \geq \ell$, we get

$$\begin{aligned} \nu^{0+}_m(\chi) = \iota_p\Bigg[C_r\left|-\frac{N(\sqrt{\mathfrak{t}\mathfrak{b}\mathfrak{c}})^2}{2}\hat{\tau}\right|^{-\frac{k+m-\mu-1-2n}{2}}\left(p^{\frac{n(n+1)}{2}}\lambda_{p,1}\cdots\lambda_{p,n}\right)^{-r} \\ \times\frac{\langle f_0, [\theta^\star_{\chi\ell}\mathcal{E}^\star_{\bar\eta}(\cdot,\frac{2m-n}{4})]|U^r_pW(Y_0)\rangle_{\mathfrak{y}_0}}{\langle f, f\rangle}\Bigg] \end{aligned} \tag{4.6.5}$$

where, recall, $\mathcal{E}^\star_{\bar\eta}(z,s) = \mathcal{E}^{\bar\eta}_{k-\frac{n}{2}-\mu}(\cdot,s)|W(Y_\chi)$.

In light of Proposition 4.5.1 we make one final artificial adjustment to this above expression by inserting a constant D_k. Define

$$D_k := \frac{i^{n\left([k-\frac{n}{2}-\mu]\right)}2^{-n(k-\mu+\frac{3}{2})}}{N(\mathfrak{b}^2\mathfrak{y}_r)^{n\left(\frac{3n-2m}{4}-k+\mu\right)}\pi^{\frac{n(m+k-n-\mu)}{2}}}\Gamma_n\left(\frac{m+k-n-\mu}{2}\right) \tag{4.6.6}$$

and the modular form

$$\mathfrak{R}^+_r(\cdot, m; \chi_\ell) := D_k\left|-\frac{N(\sqrt{\mathfrak{t}\mathfrak{b}\mathfrak{c}})^2}{2}\hat{\tau}\right|^{-\frac{k+m-\mu-1-2n}{2}}\mathbf{Pr}([\theta^\star_\chi\mathcal{E}^\star_{\bar\eta}(\cdot,\tfrac{2m-n}{4})]|U^r_p).$$

This form is an element of $\mathcal{M}_k(\Gamma[\mathfrak{b}^{-2},\mathfrak{b}^2\mathfrak{y}_\chi],\psi)$. For the values of m in the previous section it has cyclotomic Fourier coefficients that are non-zero only when $\sigma > 0$ at which point, by Proposition 4.5.1, they are

$$\begin{aligned} c(\sigma,1;\mathfrak{R}_r^+(z,m;\chi_\ell)) &= \sum_{(\sigma_1,\sigma_2)\in V_{p^r\sigma}} \chi_\ell(|\sigma_1|)|\sigma_1|^\mu \mathfrak{C}_+^\star(\sigma_2,m) P\left(\sigma_2,p^r\sigma;\tfrac{k-m-\mu}{2}\right), \\ \mathfrak{C}_+^\star(\sigma_2,m) &:= \left|-\tfrac{N(\sqrt{\mathfrak{tbc}})^2}{2}\hat{\tau}\right|^{-\frac{k+m-\mu-1-2n}{2}} |\sigma_2|^{m-n-\frac{1}{2}} \prod_{q\in\mathbf{c}} f_{\sigma,q}(\bar{\eta}(q)q^{\frac{n+\delta-1}{2}-m}). \end{aligned} \tag{4.6.7}$$

Insertion of D_k to the expression in (4.6.5) above gives

$$\nu_m^{0+}(\chi) = \iota_p\left[C_r D_k^{-1}\left(p^{\frac{n(n+1)}{2}}\lambda_{p,1}\cdots\lambda_{p,n}\right)^{-r}\frac{\langle f_0,\mathfrak{R}_r^+(\cdot,m;\chi_\ell)|W(Y_0)\rangle_{\mathfrak{y}_0}}{\langle f,f\rangle}\right]. \tag{4.6.8}$$

Proposition 4.6.4. *There exists a complex distribution ν_s^- on $\mathbb{Z}_p^\times$ which is uniquely determined on Dirichlet characters χ of p-power conductor p^{ℓ_χ} as follows. If χ is primitive with $1\le\ell_\chi\in\mathbb{Z}$, then it is defined by*

$$\begin{aligned} \nu_s^-(\chi) := {}& \frac{(-1)^{n[k]}|2\tau|^{\frac{n}{2}+\mu}}{i^d(N(\mathfrak{tbc}))^{n\mu}}\left|-\tfrac{N(\sqrt{\mathfrak{tbc}})^2}{2}\hat{\tau}\right|^{-\frac{k+3s-\mu-2-4n}{2}}\frac{p^{\ell_\chi(n+1-k-s)}G_n(\bar{\chi})}{\Lambda_\tau(s)\mathcal{G}_\tau(s)} \\ &\times\left(p^{\frac{n(n+1)}{2}}\lambda_{p,1}\cdots\lambda_{p,n}\right)^{-\ell_\chi} L_\psi(s,f,\bar{\chi}). \end{aligned} \tag{4.6.9}$$

In general, for any $\ell>\ell_\chi$ and any $r>\ell$, define

$$\begin{aligned} \nu_s^-(\chi) := {}& \frac{|\tau|^{\frac{s-n-1+k+\mu}{2}}}{c_{f_0}(\tau,1)}\left|-\tfrac{N(\sqrt{\mathfrak{tbc}})^2}{2}\hat{\tau}\right|^{-\frac{k+3s-\mu-2-4n}{2}}\Lambda_{\mathfrak{y}_0}\left(\tfrac{2s-n}{4}\right)p^{rn(\frac{3n}{2}+1-k-s)} \\ &\times\left(p^{\frac{n(n+1)}{2}}\lambda_{p,1}\cdots\lambda_{p,n}\right)^{-r} D\left(\tfrac{2s-3n-2}{4},f_0,\theta_{\chi_\ell}^\star|W(Y_\ell)\right). \end{aligned} \tag{4.6.10}$$

If $m\in\frac{1}{2}\mathbb{Z}$ with $2n+1-k+\mu\le m\le n$, then this is now normalised by letting

$$\nu_m^{0-}(\chi) := \iota_p\left[\frac{\nu_m^-(\chi)}{\pi^{c_m}\langle f,f\rangle}\right]$$

if $m\in\Omega_{n,k}^-$, and $\nu_m^{0-}(\chi)=0$ otherwise. This is a p-adic distribution since $\nu_m^-(\chi)$ has the same integral expression of (4.6.3) multiplied by factors of $|-2^{-1}N(\sqrt{t}\mathfrak{bc})^2\hat{\tau}|$, whenever it is non-zero.

Repeat the same process as before: $\nu_m^-(\chi)$ has the integral expression of (4.6.3) multiplied by factors of $|-2^{-1}N(\sqrt{t}\mathfrak{bc})^2\hat{\tau}|$ and then, after the normalisation above, use Proposition 4.3.1 when r is large enough. In the end we obtain, for $m\in\Omega_{n,k}^-$ and C_r, D_k as in (4.6.4), (4.6.6) above, the expression

$$\nu_m^{0-}(\chi) = \iota_p\left[C_r D_k^{-1}\left(p^{\frac{n(n+1)}{2}}\lambda_{p,1}\cdots\lambda_{p,n}\right)^{-r}\frac{\langle f_0,\mathfrak{R}_r^-(\cdot,m;\chi_\ell)|W(Y_0)\rangle_{\mathfrak{y}_0}}{\langle f,f\rangle}\right], \tag{4.6.11}$$

in which the holomorphic modular form

$$\mathfrak{R}_r^-(\cdot,m;\chi_\ell) := D_k\left|-\tfrac{N(\sqrt{\mathfrak{tbc}})^2}{2}\hat{\tau}\right|^{-\frac{k+3m-\mu-2-4n}{2}}\mathbf{Pr}([\theta_{\chi_\ell}^\star\mathcal{E}_{\bar{\eta}}^\star(\cdot,\tfrac{2m-n}{4})]|U_p^r)$$

has cyclotomic Fourier coefficients that are non-zero only when $\sigma > 0$. In such a case they are given, when $m \in \Omega^-_{n,k}$, by

$$\begin{aligned} c(\sigma, 1; \mathfrak{R}^-_r(\cdot, m; \chi_\ell)) &= \sum_{(\sigma_1,\sigma_2)\in V_{p^r\sigma}} \chi_\ell(|\sigma_1|)|\sigma_1|^\mu \mathfrak{C}^\star_-(\sigma_2, m) P\left(\sigma_2, p^r\sigma; \tfrac{k+m-\mu-1-2n}{2}\right), \\ \mathfrak{C}^\star_-(\sigma_2, m) :&= \left|-\tfrac{N(\sqrt{\mathfrak{t}\mathfrak{b}\mathfrak{c}})^2}{2}\hat{\tau}\right|^{-\frac{k+3m-\mu-2-4n}{2}} \prod_{q\in\mathbf{c}} f_{\sigma,q}(\bar{\eta}(q)q^{\frac{n+\delta-1}{2}-m}). \end{aligned} \tag{4.6.12}$$

Define the linear functional

$$\begin{aligned} \ell_f : \mathcal{M}_k(\Gamma[\mathfrak{b}^{-2}, \mathfrak{b}^2\mathfrak{y}_0], \psi) &\to \overline{\mathbb{Q}} \\ g &\mapsto \frac{\langle f_0, g|W(Y_0)\rangle}{\langle f, f\rangle}. \end{aligned}$$

Theorem 3.5.2 with $g = f$ gives that $\langle f, f\rangle \in \mu(\Lambda, k, \psi)\overline{\mathbb{Q}}$, and so the above linear functional indeed maps to $\overline{\mathbb{Q}}$ by Theorem 3.5.2 again.

Notice that $\ell_f \in \langle f_0, f_0\rangle\langle f, f\rangle^{-1}\mathcal{L}_f\overline{\mathbb{Q}}$, where $\mathcal{L}_f$ is defined in (3.51) of [Pan91, p. 109]; the functionals ℓ_f and $\mathcal{L}_f$ are equal up to some algebraic constant, of bounded p-adic norm, determined by the differences of the operator $W(Y_0)$ in this thesis and [Pan91]. Therefore by the property (3.52) in [Pan91, p. 109] of $\mathcal{L}_f$, there exist positive-definite matrices $\sigma_1, \dots, \sigma_t \in \mathfrak{b}^2 S^\triangledown_+$ and $\alpha_1, \dots, \alpha_t \in \mathbb{Q}(f, \Lambda, \psi)$ satisfying

$$\ell_f(g) = \sum_{i=1}^t \alpha_i c_g(\sigma_i, 1). \tag{4.6.13}$$

For any subset $\mathcal{X} \subseteq X_p^{\text{tors}}$ take the integers r and ℓ large enough so that: **(1)** all $\chi \in \mathcal{X}$ are defined modulo p^ℓ and **(2)** the expressions of (4.6.8) and (4.6.11) both hold. For any $\chi \in \mathcal{X}$ we have

$$\nu^{0\pm}_m(\chi) = \iota_p\left[C_r D_k^{-1}\left(p^{\frac{n(n+1)}{2}}\lambda_{p,1}\cdots\lambda_{p,n}\right)^{-r}\ell_f(\mathfrak{R}^\pm_r(\cdot, m; \chi_\ell))\right]. \tag{4.6.14}$$

Notice from the expressions (4.6.7) and (4.6.12) that the $\mathbb{Q}_{ab}$-coefficients of $\mathfrak{R}^\pm_r$ do not depend on the modulus p^ℓ of χ once $\ell \geq 1$; setting $c^\pm_{\sigma,r,m}(\chi) := \iota_p[c(\sigma, 1; \mathfrak{R}^\pm_r(\cdot, m; \chi_\ell)]$ therefore defines a p-adic distribution by Proposition 4.1.2. Now we have, see (4.6.13), that

$$\ell_f(\mathfrak{R}^\pm_r(\cdot, m; \chi_\ell)) = \sum_{i=1}^t \alpha_i c^\pm_{\sigma_i,r,m}(\chi), \tag{4.6.15}$$

for constants α_i of bounded p-adic valuation. Since by assumption we have

$$\left|p^{\frac{n(n+1)}{2}}\lambda_{p,1}\cdots\lambda_{p,n}\right|_p^{-r} = 1$$

and by definition C_r and D_k have bounded p-adic valuation, plus all three of these are independent of χ, we see from (4.6.14) above that whether $\nu^{0\pm}_m(\chi)$ defines a measure is directly dependent on whether the p-adic distributions $c^{\pm}_{\sigma,r,m}(\chi)$ satisfy the Kummer congruences of Proposition 4.1.5 (and therefore define p-adic measures themselves).

Now fix σ and for any $(\sigma_1,\sigma_2)\in V_{p^r\sigma}$ fix also σ_1. By definition of $c^{\pm}_{\sigma,r,m}$ we may assume $p\nmid|\sigma_1|$. We have, by Proposition 4.5.1 and the definition of $V_{p^r\sigma}$, the following congruences

$$|\sigma_2| \equiv \left|-\tfrac{N(\sqrt{\mathfrak{t}\mathfrak{b}\mathfrak{c}})^2}{2}\hat{\tau}\right| |\sigma_1|^2 \pmod{p^r\mathcal{O}_p}, \tag{4.6.16}$$

$$P(\sigma_2,p^r\sigma;\beta) \equiv |\sigma_2|^{\beta} \equiv \left[\left|-\tfrac{N(\sqrt{\mathfrak{t}\mathfrak{b}\mathfrak{c}})^2}{2}\hat{\tau}\right| |\sigma_1|^2\right]^{\beta} \pmod{p^r\mathcal{O}_p}, \tag{4.6.17}$$

where $\beta\in\mathbb{Z}$. So, recalling the Fourier expansions of (4.6.7) and (4.6.12), and using Proposition 4.5.2, we can write

$$c^{\pm}_{\sigma,r,m}(\chi) \equiv \sum_{(\sigma_1,\sigma_2)\in V_{p^r\sigma}} \chi_\ell(|\sigma_1|)|\sigma_1|^{k+m-1-2n}[\Sigma_{\sigma_2}\otimes\omega_\tau](\chi_\ell x_p^{[m]}) \pmod{p^r\mathcal{O}_p}, \tag{4.6.18}$$

where ω_τ is the primitive character associated to $\overline{\psi}^*\rho_\tau$. The conductor of ω_τ is $t\mathfrak{c}$, where t is the conductor of ρ_τ. We have $(p,t\mathfrak{c})=1$ by the following argument. Under the assumption $p\nmid\mathfrak{c}$, this is given by showing $p\nmid t$. Since $\mathfrak{b}\mathfrak{c}\subseteq 2\mathbb{Z}$ then we must have $p\nmid\mathfrak{b}^{-1}$ as well, so that $\rho_\tau(p)\neq 0$ if and only if $\left(\frac{|2N(\mathfrak{b})^{-1}\tau|}{p}\right)\neq 0$. We can assume $|\sigma_2|\not\equiv 0 \pmod{p^r\mathcal{O}_p}$, since otherwise we have $c^{\pm}_{\sigma,r,m}(\chi)\equiv 0 \pmod{p^r}$ by the congruence of (4.6.17). By definition, followed by the application of the congruence in (4.6.16) above, we have

$$\begin{aligned}\left(\frac{|2N(\mathfrak{b})^{-1}\tau|}{p}\right)^r &= \left(\frac{N(\mathfrak{b}^{-1}\mathfrak{t})}{p}\right)^{rn}\left(\frac{|\hat{\tau}|}{p}\right)^r\\ &= \left(\frac{-N(\mathfrak{b}\mathfrak{c}^2)/2}{p}\right)^{rn}\left(\frac{|\sigma_2|}{p^r}\right),\end{aligned}$$

which is non-zero by assumption and so $(t,p)=1$. Hence $\Sigma_{\sigma_2}\otimes\omega_\tau$ defines a p-adic measure. Set $c^{\pm}_{\sigma,r} := c^{\pm}_{\sigma,r,\frac{1}{2}}$, then (4.6.18) above shows that $c^{\pm}_{\sigma,r}(\chi x_p^{[m]}) = c^{\pm}_{\sigma,r,m}(\chi)$. Assume that

$$\sum_{\chi\in\mathcal{X}} b_\chi \chi x_p^{[m]} \subseteq p^N\mathcal{O}_p$$

for some $b_\chi\in\mathcal{O}_p$, then the Kummer congruences require

$$\sum_{\chi\in\mathcal{X}} b_\chi c^{\pm}_{\sigma,r}(\chi x_p^{[m]}) \in p^N\mathcal{O}_p$$

in order for $c^{\pm}_{\sigma,r}$ to define a measure. If $N\geq r$ then the congruence of (4.6.18) above gives

$$\begin{aligned}\sum_{\chi\in\mathcal{X}} b_\chi c^{\pm}_{\sigma,r}(\chi x_p^{[m]}) &\in \sum_{\chi\in\mathcal{X}} b_\chi \sum_{(\sigma_1,\sigma_2)\in V_{p^r\sigma}} \chi_\ell(|\sigma_1|)|\sigma_1|^{k+m-1-2n}[\Sigma_{\sigma_2}\otimes\omega_\tau](\chi_\ell x_p^{[m]}) + p^N\mathcal{O}_p\\ &\in \sum_{(\sigma_1,\sigma_2)\in V_{p^r\sigma}} |\sigma_1|^{k-1-2n}\sum_{\chi\in\mathcal{X}} b_\chi \int_{\mathbb{Z}_p^\times}(\chi_\ell x_p^{[m]})(y|\sigma_1|)[d\Sigma_{\sigma_2}\otimes\omega_\tau](y) + p^N\mathcal{O}_p,\end{aligned}$$

which last line is clearly in $p^N\mathcal{O}_p$, as $|\sigma_1|^{k-1-2n}\in\mathcal{O}_p^\times$ and we know $\Sigma_{\sigma_2}\otimes\omega_\tau$ is a measure satisfying the Kummer congruences by Proposition 4.5.2.

This shows that $\nu^{0\pm}_m$ defines a p-adic measure. To finish the proof define $\nu^{\pm}_f := \nu^{0\pm}_{\frac{1}{2}}$, then by the identities of (4.6.14) and (4.6.15), the congruence of (4.6.18) and the subsequent fact that $c^{\pm}_{\sigma,r}(\chi x_p^{[m]}) = c^{\pm}_{\sigma,r,m}(\chi)$, we have

$$\int_{\mathbb{Z}_p^\times} \chi x_p^{[m]} d\nu_f^{\pm} = \nu_m^{0\pm}(\chi),$$

for any m as in the statement of Theorem C. The main identities of Theorem C then follow from the primitive-character definitions, (4.6.1) and (4.6.9), of the original distributions.

4.7 Final remarks for $n = 1$

When $n = 1$, one is at a comparable advantage with respect to the calculations present in this chapter, particularly those involved with Fourier coefficients; we have already seen an example of this in Section 4.2.2, in which we used the explicit behaviour of the Fourier coefficients of an eigenform to construct the p-stabilisation. Another notable example is in the Fourier expansion of Eisenstein series. The result of this is that the definitions of the p-adic measures are more explicit, as are the methods of proof. As well, through Shimura's correspondence, this construction provides an alternative proof of the p-adic L-function for classical integral-weight modular forms and there is potential for future insights here. We discuss all of these facets further here and, for comparison, we include the statement of the main theorem for this case.

Among the Dirichlet characters of p-power conductor, the trivial character is uniquely bothersome in the construction of the measure defining the p-adic L-function. Happily, it is not necessary to define the explicit value of the measure on the trivial character since, as we mentioned in Section 4.1, any $\mathbb{C}_p$-analytic function is determined by non-trivial elements of X_p^{tors}, and we opted for this approach in this chapter. To apply the Kummer congruences, however, we still needed to involve the trivial character, and this meant lifting it up into an imprimitive non-trivial Dirichlet character and therefore the need to define the measure on imprimitive characters as well. The alternative – giving the explicit value of the measure on the trivial character – seems difficult for general $n > 1$. The issue here is twofold, on the one hand we would need to relate $L(s, f)$ with $L(s, f_0)$, as was done for non-trivial characters in Corollary 4.2.8, and on the other hand we would need to modify the Fourier coefficients of the theta series and the Eisenstein series.

To relate $L(s, f)$ with $L(s, f_0)$, when $n = 1$, we are aided by the fact that $U_p = A(p)$ and a subsequent comparison of Satake parameters gives

$$L(s, f_0) = (1 - p\lambda_{p,1}^{-1} p^{-s}) L(s, f), \tag{4.7.1}$$

this is [Mer18a, Lemma 3.2]. To explain the issue with the Fourier coefficients, recall that the product of polynomials

$$\prod_{q \in \mathbf{c}} f_{\sigma,q}\left(\chi(q) q^{\frac{n+\delta-1}{2} - m}\right), \tag{4.7.2}$$

occurring in the Fourier expansion of the Eisenstein series $\mathcal{E}^{\star}_{\chi}(z, \frac{2m-n}{4})$ of Proposition 4.5.1, define a p-adic measure whenever χ has p-power conductor (Proposition 4.5.2). If χ is trivial then the character appearing in the Eisenstein series is the principal character modulo $\mathfrak{c}_0$ – so the above product would still define a p-adic measure – but the character in the theta series remains the trivial character and we end up with slightly different Fourier coefficients in this case. Therefore we would need an explicit way of comparing the Fourier coefficients for non-trivial characters with those for the trivial character in order to use them in the Kummer congruences. This is likely possible by considering the Dirichlet series

$$\alpha_{\mathfrak{c}}^{1-\delta}(N(\mathfrak{b})^{\delta}\sigma, m - \tfrac{n}{2}, \chi), \tag{4.7.3}$$

appearing on the left-hand side of the identity (16.46) of [Shi00] between this series and the product of polynomials in (4.7.2) above, and by trying to generalise an identity we have in (4.7.4) below for $n = 1$. In the prenominate special case, the key ingredient of the Fourier coefficients, at $1 \leq t \in \mathbb{Z}$, of the Eisenstein series is the divisor sum

$$\sigma_{m-\frac{3}{2},\star}(t, \chi) := \sum_{d|t} \chi\left(\tfrac{t}{d}\right) d^{m-\frac{3}{2}},$$

which is essentially the Dirichlet series α in (4.7.3) above. That the p-modified divisor sum

$$\sigma^{(p)}_{m-\frac{3}{2},\star}(t, \chi) := \sum_{p\nmid d|t} \chi\left(\tfrac{t}{d}\right) d^{m-\frac{3}{2}}$$

defines a p-adic measure, is well known – see, for example, Chapter 7 of [Hid93] – and clearly if $\chi \in X_p^{\mathrm{tors}}$ is non-trivial we have that these two divisor sums are equal. Moreover, by an elementary calculation, we showed in [Mer18a, Lemma 6.4] that

$$\chi(p)(\chi(p) + p^{m-\frac{3}{2}})\sigma^{(p)}_{m-\frac{3}{2},\star}(t, \chi) = \sigma_{m-\frac{3}{2},\star}(p^2 t, \chi) - p^{2m-3}\sigma_{m-\frac{3}{2},\star}(t, \chi), \tag{4.7.4}$$

which is a generalisation of the fact that, in the case $\kappa \in \mathbb{Z}$ and $E_{\kappa}(z) := E_{\kappa}(z, \frac{\kappa}{2}; 1, SL_2(\mathbb{Z}))$, we have $E_{\kappa}(z) - p^{\kappa-1}E_{\kappa}(pz)$ is a modular form whose coefficients define p-adic measures (i.e. it is a p-adic modular form, see Chapter 7 of [Hid93]). For our Eisenstein series of *principal* character χ_0 mod p the identity (4.7.4) gives

$$0 = \sigma_{m-\frac{3}{2},\star}(p^2 t, \chi_0) - p^{2m-3}\sigma_{m-\frac{3}{2},\star}(t, \chi_0), \tag{4.7.5}$$

and a version of this identity allows one to match up exactly the Fourier coefficients of $\mathbf{Pr}(\theta_{\chi}\mathcal{E}^{\star}_{\chi}(z, \frac{2m-n}{4})|U_p^r)$, when χ is non-trivial, to those of

$$\mathbf{Pr}(\theta\mathcal{E}^{\star}_{\chi_0}(z, \tfrac{2m-n}{4})|[p^{3-2m}U_p^{r+1} - p^{k-m-\mu}U_p^r]). \tag{4.7.6}$$

See later or [Mer18a, Section 6] for further details. In order to obtain the operator $U_p^{r+1} - U_p$ in the above equation, we use the fact that $p\lambda_{p,1}f_0 = f_0|U_p$ and therefore include a factor of the form $(p\lambda_{p,1} - 1)$ in the definition of the measure on the trivial character. Upon manipulation of the inner product via the trace operator, as in Section 4.3.1, with the fact that U_p is self-adjoint one obtains a form as in (4.7.6) above.

In [Mer18a] we focused only on the set $\Omega^+_{1,k}\backslash\{\frac{3}{2}\}$ of special values and we produced, for an eigenform $f \in \mathcal{S}_k(\Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}])$, measures $\nu^{(\mu)}_{f,m}$ which are only non-zero for characters of certain parity. These are the p-adic distributions ν^{0+}_m we saw in the general $n \geq 1$ case. For $n = 1$, we can see by the definition of (4.2.9) that $f_1 \neq 0$ if $f \neq 0$, so we do not need to make the further assumption that $c_{f_1}(\tau) \neq 0$ here.

Theorem 4.7.1 ([Mer18a], Theorem 6.1)**.** *Let $\mu \in \{0, 1\}$ and $m \in \frac{1}{2}\mathbb{Z}$ satisfy $\frac{3}{2} < m \leq k$ and $m - k + \mu \in 2\mathbb{Z}$. Assume $p \nmid \mathfrak{c}$ and $f \in \mathcal{S}_k(\Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}])$ is a p-ordinary eigenform; take $1 \leq \tau \in \mathbb{Z}$ such that $c_f(\tau) = 1$. There exists a constant D_μ and p-adic measures $\nu^{(\mu)}_{f,m}$ on $\mathbb{Z}_p^\times$ such that if χ is a Dirichlet character, of conductor p^{ℓ_χ} with $1 \leq \ell_\chi \in \mathbb{Z}$, and $\chi(-1) = (-1)^{[k]+\mu}$ then*

$$\int_{\mathbb{Z}_p^\times} \chi d\nu^{(\mu)}_{f,m} = D_\mu p^{2k-2m-1} p^{\ell_\chi(m-1)} \frac{G(\bar{\chi})}{\Lambda_\tau(m)} (p\lambda_{p,1})^{-\ell_\chi} \frac{L(m-\mu, f, \bar{\chi})}{\pi^{m+k-\mu-1}\langle f, f\rangle};$$

if $\chi(-1) \neq (-1)^{[k]+\mu}$ then $\int_{\mathbb{Z}_p^\times} \chi d\nu^{(\mu)}_{f,m} = 0$. If $\mu \equiv [k] \pmod 2$ then

$$\int_{\mathbb{Z}_p^\times} d\nu^{(\mu)}_{f,m} = D_\mu (1 - p\lambda_{p,1}^{-1} p^{\mu-m}) \frac{p\lambda_{p,1} - p^{m+\mu-1}}{p^{2m-3}(p\lambda_{p,1})} \Lambda_\tau(m)^{-1} \frac{L(m-\mu, f)}{\pi^{m+k-\mu-1}\langle f, f\rangle};$$

if $\mu \not\equiv [k] \pmod 2$ then $\int_{\mathbb{Z}_p^\times} d\nu^{(\mu)}_{f,m} = 0$.

To define the p-adic L-function we put $\nu^{(\mu)}_f := \nu^{(\mu)}_{f,k-\mu}$. We have $d\nu^{(\mu)}_{f,m}(x) = x^{m-k+\mu} d\nu^{(\mu)}_{f,k-\mu}$ – this is [Mer18a, Proposition 6.12] – and set

$$\mathcal{L}_p(s, f, \chi) := \int_{\mathbb{Z}_p^\times} \chi x_p^{s-k+\mu} d\mu^{(\mu)}_f.$$

The proof of Theorem 4.7.1 is structurally similar to that of the general $n \geq 1$ case of this chapter. Begin by producing the p-stabilisation f_1 of Section 4.2.2, then since $U_p = A(p)$ we have $f_1|A(p^m) = (p\lambda_{p,1})^m f_1$ and note

$$\sum_{m=0}^{\infty} (p\lambda_{p,1})^m t^m = (1 - p\lambda_{p,1}t)^{-1}.$$

Through the association of Satake parameters in (2.2.11), we see that the f_1 has the same parameters as f, and therefore the identity of (4.7.1) between $L(s, f_1)$ and $L(s, f)$ follows from Definition 2.2.5.

Next, one proves the analogous results of Section 4.3.1. The transformation formula of the theta series and Fourier expansions of Eisenstein series are well known in this setting.

Instead of appealing to the abstract Kummer congruences, the proof of Theorem 4.7.1 above is completed by showing that the measure of any arbitrary open subgroup $e + p^r\mathbb{Z}_p$, where e is an integer coprime to p and $0 \leq r \in \mathbb{Z}$, is of bounded p-adic valuation as e and r range. To do this we show explicit congruences for the Fourier coefficients of the theta series and Eisenstein series. Let C_r denote the set of all Dirichlet characters whose conductors divide p^r, then we have that

$$\nu_{f,m}^{(\mu)}(e+p^r\mathbb{Z}_p) = \varphi(p^r)^{-1}\sum_{\chi\in C_r}\chi(e^{-1})\int_{\mathbb{Z}_p^\times}\chi d\nu_{f,m}^{(\mu)},$$

where φ is Euler's totient function. Thus, by definition of the measure, we relate the above to $\ell_f(\mathfrak{R}'_r)$, where $\mathfrak{R}'_r$ is $\varphi(p^r)^{-1}$ multiplied by a sum over all $\chi \in C_r$ of the projected theta series and Eisenstein series. To show that $c(t;\mathfrak{R}'_r) \in \mathbb{Z}_p \cap \mathbb{Q}$ are all p-integral thus involves proving specific congruences modulo p^r. The inclusion of the factor

$$\frac{p\lambda_{p,1} - p^{m+\mu-1}}{p^{2m-3}(p\lambda_{p,1})}$$

in the definition of the measure and subsequently using the identity in (4.7.5) removes the inherent troubles with the trivial character that we discussed above – see [Mer18a, Lemma 6.10] for the details here. The space $\mathcal{M}_k(\Gamma[\mathfrak{b}^{-2},\mathfrak{b}^2\mathfrak{c}_0],\mathbb{Z}_p\cap\mathbb{Q})\otimes\mathbb{Z}_p$ is a finitely-generated $\mathbb{Z}_p$-module, say with generators g_j, and therefore we have

$$|\ell_f(\mathfrak{R}'_r)|_p \leq \max_j\{|\ell_f(g_j)|_p\},$$

for any r.

A final point of interest from this construction is in the determination of the integrality of the measure produced. The measure is integral when it takes values in $\mathbb{Z}_p$. As a result of the Shimura correspondence any differences in determining integrality of the measure in our setting then offers up some alternative insights into determining the integrality of the original p-adic measure for the L-function of integral-weight modular forms. The periods appearing in the denominator of the measure are naturally pivotal to the integrality of the measures and our construction here differs significantly with that found in [Hid93], which uses the Eichler-Shimura isomorphism and modular symbols. In that construction, integrality is determined via congruences between cusp forms and Eisenstein series.

Our construction is much closer in line with the p-adic measure for the adjoint square L-function of modular forms of integral weight, as seen in [CS86, Hid00], in which $\langle f, f\rangle$ plays the role of the period. In the construction of the p-adic adjoint square, questions of integrality are settled through the congruence module, as seen in [Hid00, p. 296]. The potential upshot of this is that integrality for the p-adic measure constructed here is likely to be determined through the congruence module which would involve congruences between cusp forms of half-integral weight and which provides alternative means for approaching the integrality of the p-adic L-function for integral-weight modular forms.

Chapter 5

L-functions for vector-valued modular forms

This chapter consists of joint work with Thanasis Bouganis, the results of which can also be found in [BM18].

The focus of this chapter is on a further abstraction on the kinds of modular forms one can consider. Treating now both integral and half-integral weight forms at once, we vary their range to have values in some complex vector space V; the weights are further adjusted by the addition of representations ρ over V. Forms in the case, considered previously, of $V = \mathbb{C}$ and $\rho = 1$ are called *scalar-valued* (or just *scalar*) modular forms, else they are known as *vector-valued* modular forms.

As in the scalar-weight case, the standard L-function of a vector-valued eigenform can be defined; the principal outcome of this joint work is the establishment of the Rankin-Selberg integral expression and of fundamental analytic properties akin to Theorem 2.2.6 of this thesis and Theorem A of [Shi96]. These results are given with relatively high generality. One of these analytic properties is a partial result on improving the half-plane of non-vanishing for the L-function – i.e. for $\mathfrak{Re}(s) > \frac{3n}{2} + 1$. The importance of this has already been illustrated in this thesis in the definition of the constants $\mu'(\Lambda, k, \psi)$ and $\mu(\Lambda, k, \psi)$ of (3.1.2) and (3.1.4) in Chapter 3, which ultimately determined the bounds on k in the main theorems of that chapter.

The key to the method here is the use of vector-valued theta series and their cuspidality, given in Section 5.2 after an expository Section 5.1. The rest of this chapter establishes the Rankin-Selberg integral expression in a similar vein to Section 2.4, and is finished by the statement and proofs of the main theorems.

Algebraic properties of this L-function have been studied before in specific cases, for example by Böcherer, Takei, and Kozima in [Böc85, Tak92, Koz00] respectively, in which the $\rho = \mathrm{Sym}^\ell$ representation was considered and the doubling method employed. The most general results can be seen in the work of Piatetski-Shapiro and Rallis in [PSR87] and [PSR88], which both use the theory and language of automorphic representations, the former using the doubling method and the latter the Rankin-Selberg method. The L-functions in [PSR87, PSR88] are untwisted, have Euler factors removed, and the expressions provided are non-explicit. This section avoids these latter issues and considers a different class of representations to Sym^ℓ. It is hoped that the work here could lead to algebraicity results beyond those given by the doubling method, which results would be a consequence of a full result improving the range of non-vanishing of the L-function.

5.1 Vector-valued modular forms

Notation. $k \in \frac{1}{2}\mathbb{Z}$ – integral or half-integral weight.
$[k] = k$ if $k \in \mathbb{Z}$; $[k] = k - \frac{1}{2}$ if $k \notin \mathbb{Z}$.
$\mathfrak{b}$ – fractional ideal of $\mathbb{Q}$; $\mathfrak{c}$ – integral ideal of $\mathbb{Z}$.
$(\mathfrak{b}^{-1}, \mathfrak{bc}) \subseteq 2\mathbb{Z} \times 2\mathbb{Z}$ *if* $k \notin \mathbb{Z}$.
$\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}]$; $D = D[\mathfrak{b}^{-1}, \mathfrak{bc}]$.
$P = \{\alpha \in Sp_n(\mathbb{Q}) \mid c_\alpha = 0\}$; $r_P : P_\mathbb{A} \to M_\mathbb{A}$ – lift.
$\mu(\alpha, z) = c_{\alpha_\infty} z + d_{\alpha_\infty}$ if $\alpha \in G_\mathbb{A}$ and $z \in \mathbb{H}_n$.
$\alpha \cdot z = (a_{\alpha_\infty} z + b_{\alpha_\infty})\mu(\alpha, z)^{-1}$ if $\alpha \in G_\mathbb{A}$ and $z \in \mathbb{H}_n$.
$S_+ = \{\xi \in M_n(\mathbb{Q}) \mid \xi^T = \xi, \xi \geq 0\}$.
$S^\triangledown$ – set of symmetric half-integral $n \times n$ matrices.
$S^\triangledown_\mathfrak{b}$ – set of symmetric $n \times n$ matrices τ such that $\tau \in N(\mathfrak{b})S^\triangledown$.
$S(\mathfrak{b}^{-1}) = \{\xi \in M_n(\mathfrak{b}^{-1}) \mid \xi^T = \xi\}$; $S_\mathbf{f}(\mathfrak{b}^{-1}) = \prod_p S(\mathfrak{b}_p^{-1})$.
$\tilde{q} = (q^T)^{-1}$ for any invertible matrix q.
$\mathsf{X}_p = M_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p)$; $\mathsf{X} = GL_n(\mathbb{Q})_\mathbf{f} \cap \prod_p \mathsf{X}_p$.
$Z_0 = \{\mathrm{diag}[\tilde{q}, q] \mid q \in \mathsf{X}\}$; $Z = D[2,2]Z_0D[2,2]$; $\mathfrak{Z} = \mathrm{pr}^{-1}(Z)$.

Let V denote a finite-dimensional complex vector space and let

$$\rho : GL_n(\mathbb{C}) \to GL(V) \tag{5.1.1}$$

be a rational representation – i.e. it is a rational map of algebraic varieties. Let $\xi \in M_\mathbb{A}$ be such that $\mathrm{pr}(\xi) = \alpha \in G_\mathbb{A}$, then from any function $f : \mathbb{H}_n \to V$ we can define a new function $f|_\rho \xi : \mathbb{H}_n \to V$ by

$$(f|_\rho \xi)(z) := \rho(\mu(\alpha, z))^{-1} f(\alpha \cdot z).$$

The kinds of modular forms we will be considering transform with respect to representations of the form $\rho_k := h^{k-[k]} \otimes \det^{[k]} \otimes \rho$, where ρ is as in (5.1.1); $k \in \frac{1}{2}\mathbb{Z}$ is now an integral or half-integral weight; $\det : GL_n(\mathbb{C}) \to GL(V)$ is the scalar determinant representation; and $h : GL_n(\mathbb{C}) \to GL(V)$ is a scalar representation satisfying

$$h(\mu(\sigma, z)^{-1})(v) = h_\sigma(z)^{-1} v,$$
$$h(\mu(\alpha, z)^{-1})(v) = J^{\frac{1}{2}}(\alpha, z) v,$$

for any $\sigma \in \mathfrak{M}$, $\alpha \in \mathfrak{Z}$, and $v \in V$.

The congruence subgroups $\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}]$ for a fractional ideal $\mathfrak{b}$ and integral ideal $\mathfrak{c}$ are once again considered, but note that the conditions $2 \mid \mathfrak{b}^{-1}$ and $2 \mid \mathfrak{bc}$ are only imposed in the case $k \notin \mathbb{Z}$.

Definition 5.1.1. Let $\rho : GL_n(\mathbb{C}) \to GL(V)$ be a rational representation, let k be an integral or half-integral weight, and let $\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}]$ be a congruence subgroup (contained in $\mathfrak{M}$ if $k \notin \mathbb{Z}$). A *vector valued modular form of weight* ρ_k *and level* Γ is a function $f : \mathbb{H}_n \to V$ that is holomorphic on $\mathbb{H}_n$ (and at cusps if $n = 1$) and that satisfies $f|_{\rho_k}\alpha = f$ for all $\alpha \in \Gamma$. Holomorphicity is understood component-wise with respect to some finite fixed basis of V over $\mathbb{C}$. The space of such modular forms is denoted $\mathcal{M}_{\rho_k}(\Gamma)$.

If $f \in \mathcal{M}_{\rho_k}(\Gamma)$ and $k \in \mathbb{Z}$ take $\alpha \in G$, otherwise take $\alpha \in \mathfrak{M}$, then $f|_{\rho_k}\alpha$ has a Fourier expansion of the form

$$(f|_{\rho_k}\alpha)(z) = \sum_{0 \le \tau \in S_{\mathfrak{b}}^{\triangledown}} c_\alpha(\tau) e(\mathrm{tr}(\tau z)),$$

where $c_\alpha(\tau) \in V$. The subspace of cusp forms, denoted $\mathcal{S}_{\rho_k}(\Gamma)$, is defined as all those $f \in \mathcal{M}_{\rho_k}(\Gamma)$ which have $c_\alpha(\tau) = 0$, for all such α and for all $0 \le \tau \in S_{\mathfrak{b}}^{\triangledown}$ such that $\det(\tau) = 0$. Write $\mathcal{X}_{\rho_k} := \bigcup_\Gamma \mathcal{X}_{\rho_k}(\Gamma)$ for any $\mathcal{X} \in \{\mathcal{M}, \mathcal{S}\}$, where the union is over all congruence subgroups (contained in $\mathfrak{M}$ if $k \notin \mathbb{Z}$).

Take, as usual, a Hecke character ψ such that $\psi_\infty(x)^n = \mathrm{sgn}(x_\infty)^{n[k]}$ and $\psi_p(a) = 1$ for any $a \in \mathbb{Z}_p^\times$ such that $a \in 1 + \mathfrak{c}_p$, and define the spaces

$$\begin{aligned} \mathcal{M}_{\rho_k}(\Gamma, \psi) &:= \{f \in \mathcal{M}_{\rho_k} \mid f|_{\rho_k}\gamma = \psi_{\mathfrak{c}}(|a_\gamma|)f \text{ for all } \gamma \in \Gamma\}, \\ \mathcal{S}_{\rho_k}(\Gamma, \psi) &:= \mathcal{M}_{\rho_k}(\Gamma, \psi) \cap \mathcal{S}_{\rho_k}. \end{aligned}$$

We understand $\mathrm{pr}_{\mathbb{A}}$, pr_p, r, r_P, and r_Ω to be the identity maps on $G_{\mathbb{A}}$, $Sp_n(\mathbb{Q}_p)$, G, $P_{\mathbb{A}}$, and Ω respectively if $k \in \mathbb{Z}$. By strong approximation we have that $G\,\mathrm{pr}^{-1}(D[\mathfrak{b}^{-1}, \mathfrak{bc}])$ is $G_{\mathbb{A}}$ if $k \in \mathbb{Z}$ and is $M_{\mathbb{A}}$ if $k \notin \mathbb{Z}$. If $f \in \mathcal{M}_{\rho_k}(\Gamma, \psi)$ then its adelisation is a map $f_{\mathbb{A}} : \mathrm{pr}^{-1}(G_{\mathbb{A}}) \to V$ defined by

$$f_{\mathbb{A}}(x) := \psi_{\mathfrak{c}}(|d_w|)(f|_{\rho_k}w)(\mathbf{i}),$$

where $x = \alpha w$ for $\alpha \in G$ and $w \in \mathrm{pr}^{-1}(D[\mathfrak{b}^{-1}, \mathfrak{bc}])$, and $\mathbf{i} = iI_n$. Analogously to [Shi95b, (1.16)], one can check by definition that if $x \in \mathrm{pr}^{-1}(G_{\mathbb{A}})$, $\alpha \in G$ and $w \in \mathrm{pr}^{-1}(D[\mathfrak{b}^{-1}, \mathfrak{bc}])$ is such that $w \cdot \mathbf{i} = \mathbf{i}$, then

$$f_{\mathbb{A}}(\alpha x w) = \psi_{\mathfrak{c}}(|d_w|)\rho_k(\mu(w, \mathbf{i})^{-1})f_{\mathbb{A}}(x). \tag{5.1.2}$$

Let $t \in \mathrm{pr}^{-1}(G_{\mathbf{f}})$, set both $\Gamma^t := G \cap \mathrm{pr}(t)D[\mathfrak{b}^{-1}, \mathfrak{bc}]\,\mathrm{pr}(t)^{-1}$ and

$$M_{\rho_k}(\Gamma^t, \psi) := \{f \in \mathcal{M}_{\rho_k} \mid f|_{\rho_k}\gamma = \psi_{\mathfrak{c}}(|a_{t^{-1}\gamma t}|)f \text{ for any } \gamma \in \Gamma^t\}.$$

Proposition 5.1.2. *For any $t \in \mathrm{pr}^{-1}(G_{\mathbf{f}})$ and $y \in G_\infty$ we have $f_{\mathbb{A}}(ty) = (f_t|_{\rho_k}y)(\mathbf{i})$ for some $f_t \in \mathcal{M}_{\rho_k}(\Gamma^t, \psi)$.*

Proof. Heuristically, the function f_t is the translation of f to some cusp. Decompose $ty = \alpha_t w_t$ according to $G\,\mathrm{pr}^{-1}(D[\mathfrak{b}^{-1}, \mathfrak{bc}])$; by definition

$$f_{\mathbb{A}}(ty) = \psi_{\mathfrak{c}}(|d_{w_t}|)(f|_{\rho_k} w_t)(\mathbf{i}),$$

and so define the function f_t by letting $y \in G_{\mathbb{A}}$ vary in

$$f_t(y \cdot \mathbf{i}) := \psi_{\mathfrak{c}}(|d_{w_t}|)\rho_k(\mu(y, \mathbf{i}))(f|_{\rho_k} w_t)(\mathbf{i}).$$

□

The above result is an analogue of [Shi00, (20.3b)].

Theorem 5.1.3. *If $f \in \mathcal{M}_{\rho_k}(\Gamma, \psi)$, then for any $\tau \in S_+$ and $q \in GL_n(\mathbb{A}_{\mathbb{Q}})$ there exists $c_f(\tau, q) \in V$ such that $f_{\mathbb{A}}$ has the following Fourier expansion*

$$f_{\mathbb{A}}\left(r_P \begin{pmatrix} q & s\tilde{q} \\ 0 & \tilde{q} \end{pmatrix}\right) = \rho_{[k]}(q_\infty^T)\|q_\infty\|^{k-[k]} \sum_{\tau \in S_+} c_f(\tau, q) e_\infty(\mathrm{tr}(iq^T\tau q)) e_{\mathbb{A}}(\mathrm{tr}(\tau s)), \tag{5.1.3}$$

for any $s \in S_{\mathbb{A}}$.

The coefficients obey:

(i) $c_f(\tau, q) \neq 0$ *only if* $e_{\mathbf{f}}(\mathrm{tr}(q^T\tau q s)) = 1$ *for all* $s \in S_{\mathbf{f}}(\mathfrak{b}^{-1})$*;*

(ii) $c_f(\tau, q) = c_f(\tau, q_{\mathbf{f}})$*;*

(iii) $c_f(b^T\tau b) = \rho_{[k]}(b^T)\|b\|^{k-[k]} c_f(\tau, bq)$ *for any* $b \in GL_n(\mathbb{Q})$*;*

(iv) $\psi_{\mathbf{f}}(|a|)c_f(\tau, qa) = c_f(\tau, q)$ *for any* $\mathrm{diag}[a, \tilde{a}] \in D[\mathfrak{b}^{-1}, \mathfrak{bc}]$*;*

(v) if $\beta \in G \cap \mathrm{diag}[r, \tilde{r}]D$ *and* $r \in GL_n(\mathbb{A}_{\mathbb{Q}})_{\mathbf{f}}$*, then*

$$\rho_k(\mu(\beta, \beta^{-1}z)) f(\beta^{-1}z) = \psi_{\mathfrak{c}}(|d_\beta r|) \sum_{\tau \in S_+} c_f(\tau, r) e_\infty(\mathrm{tr}(\tau z)).$$

Proof. The proof of this theorem, that we detail regardless, is almost exactly that of Proposition 20.2 in [Shi00]. Let $x = r_P \left(\begin{smallmatrix} q & s\tilde{q} \\ 0 & \tilde{q} \end{smallmatrix}\right)$ and put $t := x_{\mathbf{f}}$. For each such q and s, the functions $f_t \in \mathcal{M}_{\rho_k}(\Gamma^t, \psi)$ given in Proposition 5.1.2 have Fourier expansions

$$f_t(z) = \sum_{\tau \in S_+} c'_f(\tau) e_\infty(\mathrm{tr}(\tau z)),$$

where the coefficients $c'_f(\tau) = c'_f(\tau, q, s)$ depend on q and s. Note $x_\infty \mathbf{i} = q_\infty^T q_\infty i + s_\infty$, $x = t x_\infty$, and so by Proposition 5.1.2 we have got

$$f_{\mathbb{A}}(x) = (f_t|_{\rho_k} x)(\mathbf{i}) = \rho_{[k]}(q_\infty^T)\|q_\infty\|^{k-[k]} \sum_{\tau \in S_+} c'_f(\tau, q, s) e_\infty(\mathrm{tr}(iq^T\tau q)) e_\infty(\mathrm{tr}(\tau s)),$$

where we used that $\mathrm{pr}(x) \in P_{\mathbb{A}}$ and the behaviour of the factor of automorphy h on parabolic elements, see 2.1.2. Defining $c_f(\tau, q, s) := e_{\mathbf{f}}(-\operatorname{tr}(\tau s))c'_f(\tau, q, s)$ gives

$$f_{\mathbb{A}}(x) = \rho_{[k]}(q_\infty^T)\|q_\infty\|^{k-[k]} \sum_{\tau \in S_+} c_f(\tau, q, s)e_\infty(\operatorname{tr}(iq^T\tau q))e_{\mathbb{A}}(\operatorname{tr}(\tau s)).$$

Since $f_{\mathbb{A}}(\alpha x w) = f_{\mathbb{A}}(x)$ for any $\alpha = \left(\begin{smallmatrix} I_n & s' \\ 0 & I_n \end{smallmatrix}\right) \in G$ and any $w = \left(\begin{smallmatrix} I_n & s'' \\ 0 & I_n \end{smallmatrix}\right) \in D[\mathfrak{b}^{-1}, \mathfrak{bc}]_{\mathbf{f}}$ (note $w_\infty = I_{2n}$, so $w \cdot \mathbf{i} = \mathbf{i}$), we have $c_f(\tau, q, s) = c_f(\tau, q, s + s' + qs''q^T)$ for any $s' \in S$ and $qs''q^T \in \prod_p q_p M_n(\mathfrak{b}_p^{-1})q_p^T$. Each place of this latter product is a $\mathbb{Z}_p$-lattice in S_p, so we get independence of the $c_f(\tau, q, s)$ on $s \in S_{\mathbb{A}}$, and therefore the desired Fourier expansion.

Properties (i) – (iv) are easily deduced from the transformation formula of $f_{\mathbb{A}}$ in (5.1.2) above. To see property (v) let $y \in P_\infty$ satisfy $z = y \cdot \mathbf{i}$. By definition of β and r we have $\beta^{-1} \operatorname{diag}[r, \tilde{r}]y =: w \in D$ (note that y is trivial at the finite places), and by the transformation of (5.1.2) we have $f_{\mathbb{A}}(\operatorname{diag}[r, \tilde{r}]y) = f_{\mathbb{A}}(\beta \operatorname{diag}[r, \tilde{r}]y) = f_{\mathbb{A}}(w)$. We get

$$f_{\mathbb{A}}(w) = \psi_{\mathfrak{c}}(|d_w|)(f|_{\rho_k}\beta^{-1}y)(\mathbf{i}) = \psi_{\mathfrak{c}}(|d_w|)\rho_{[k]}(d_y)^{-1}\|d_y\|^{[k]-k}(f|_{\rho_k}\beta^{-1})(z),$$

using that $w_\infty = \beta^{-1}y$. Applying the cocycle relation $\mu(\beta^{-1}, z)^{-1} = \mu(\beta, \beta^{-1}z)$ gives

$$\rho_k(\mu(\beta, \beta^{-1}z))f(\beta^{-1}z) = \psi_{\mathfrak{c}}^{-1}(|d_w|)\rho_{[k]}(d_y)\|d_y\|^{k-[k]}f_{\mathbb{A}}(\operatorname{diag}[r, \tilde{r}]y).$$

To finish (v), note that $(d_w)_p = d_\beta^{-1}\tilde{r}_p$ for any prime p, and apply the Fourier expansion of (5.1.3) and property (ii) in the statement of the theorem to $f_{\mathbb{A}}(\operatorname{diag}[r, \tilde{r}]y)$ – i.e. with $q = a_y r$ and $s = b_y d_y^{-1}$ – noting also that $d_y a_y^T = I_n$. $\square$

We endow V with a Hermitian inner product $\prec \cdot, \cdot \succ$ with respect to which ρ behaves as

$$\prec \rho(M)\cdot, \cdot \succ = \prec \cdot, \rho(\bar{M}^T)\cdot \succ,$$

for any $M \in GL_n(\mathbb{C})$. Suppose that $f, g : \mathbb{H}_n \to V$ satisfy $f|_{\rho_k}\gamma = \psi_{\mathfrak{c}}(|a_\gamma|)f$ and $g|_{\rho_k}\gamma = \psi_{\mathfrak{c}}(|a_\gamma|)g$ for all $\gamma \in \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}]$. For any $y \in Y$ let $\sqrt{y}$ denote an element of $M_n(\mathbb{R})$ such that $(\sqrt{y})^2 = y$, then the Petersson inner product of f and g is given by

$$\langle f, g \rangle := \operatorname{Vol}(\Gamma\backslash\mathbb{H}_n)^{-1} \int_{\Gamma\backslash\mathbb{H}_n} \prec \rho_k(\sqrt{y})f, \rho_k(\sqrt{y})g \succ d^\times z,$$

whenever this integral is convergent.

This section is now ended with a brief discussion on Hecke operators in this setting and how the L-functions are defined. Essentially, it is the same.

Hecke operators are defined using the exact same abstract Hecke ring as in the scalar case – see Section 2.2 for the half-integral weight case. For the integral-weight case the Hecke ring is $\mathcal{R}(D, Z')$, where $Z' = DZ_0D$ – as it is in the scalar case. The only thing that needs modifying is the representation of this ring on the space of vector-valued modular forms in $\mathcal{M}_{\rho_k}(\Gamma, \psi)$, and this is done predictably. If $k \notin \mathbb{Z}$ then the action of $T = \widehat{\mathfrak{D}}(\alpha, t)\widehat{\mathfrak{D}}$ on $f_{\mathbb{A}}$ is exactly as it is in (2.2.4); if $k \in \mathbb{Z}$ then the action of $T = D\alpha D = \bigsqcup_\beta D\beta$, with $\alpha, \beta \in Z$, is given by

$$(f_{\mathbb{A}}|T_\psi)(x) := \sum_{\beta} \psi_{\mathfrak{c}}(|a_\beta|)^{-1} f_{\mathbb{A}}(x\beta^{-1}).$$

Globally, the integral and half-integral weight actions can be written in the same way as each other using the slash operator $|_{\rho_k}$. With $D = D[\mathfrak{b}^{-1}, \mathfrak{bc}]$, $q \in \mathsf{X}$, and

$$G \cap (D\,\mathrm{diag}[\tilde{q}, q]D) = \Gamma\beta\Gamma = \bigsqcup_{\alpha} \Gamma\alpha,$$

for $\alpha, \beta \in G \cap Z$, the action of this on $f \in \mathcal{M}_{\rho_k}(\Gamma, \psi)$ is given as

$$(f|T_{q,\psi})(z) := \sum_{\alpha} \psi_{\mathfrak{c}}(|a_\alpha|)^{-1} (f|_{\rho_k}\alpha)(z).$$

The Satake map is defined on the abstract level so the association of Satake parameters to a non-zero eigenform, and therefore the definition of the standard L-function, remains the same. The case $k \notin \mathbb{Z}$ is given by Definition 2.2.5, and the $k \in \mathbb{Z}$ case is quickly given here. If $\Lambda(p^m) = \Lambda(A(p^m)) \in \mathbb{C}$ denote the eigenvalues of f, then by [Shi94, p. 554] there exists $(\lambda_{p,1}, \dots, \lambda_{p,n}) \in \mathbb{C}^n$ such that

$$\sum_{m=0}^{\infty} \Lambda(p^m) t^m = \begin{cases} \displaystyle\prod_{i=1}^{n} (1 - p^n \lambda_{p,i} t)^{-1} & \text{if } p \mid \mathfrak{c}, \\ \displaystyle\frac{1-t}{1-p^n t} \prod_{i=1}^{n} \frac{1 - p^{2i} t^2}{(1 - p^n \lambda_{p,i} t)(1 - p^n \lambda_{p,i}^{-1} t)} & \text{if } p \nmid \mathfrak{c}, \end{cases} \tag{5.1.4}$$

which, note, differs slightly from the corresponding expression of the half-integral weight case in (2.2.10). The L-function of f twisted by a Hecke character χ is then defined, in the same way as for $k \notin \mathbb{Z}$, by

$$L_p(t) := \begin{cases} \displaystyle\prod_{i=1}^{n} (1 - p^n \lambda_{p,i} t) & \text{if } p \mid \mathfrak{c}, \\ \displaystyle(1 - p^n t) \prod_{i=1}^{n} (1 - p^n \lambda_{p,i} t)(1 - p^n \lambda_{p,i}^{-1} t) & \text{if } p \nmid \mathfrak{c}, \end{cases} \tag{5.1.5}$$

$$L_\psi(s, f, \chi) := \prod_p L_p\left((\psi^{\mathfrak{c}} \chi^*)(p) p^{-s}\right)^{-1}.$$

5.2 Vector-valued theta series

Notation. $k \in \frac{1}{2}\mathbb{Z}$ – integral or half-integral weight.
$[k] = k$ if $k \in \mathbb{Z}$; $[k] = k - \frac{1}{2}$ if $k \notin \mathbb{Z}$.
V – finite-dimensional complex vector space.
$\rho : GL_n(\mathbb{C}) \to GL(V)$ – rational representation; $\rho_k = h^{k-[k]} \otimes \det^{[k]} \otimes \rho$.
$\widetilde{Q} = (Q^T)^{-1}$ for any invertible matrix Q.
$\tau[Q] = Q^T \tau Q$ for any $n \times n$ matrices τ and Q.
$S_+ = \{\xi \in M_n(\mathbb{Q}) \mid \xi^T = \xi, \xi \geq 0\}$.
$d = \frac{n^2}{2}$ if n is even, $d = 0$ if n is odd.

In the scalar-valued case we saw how, alongside the eigenform f, the key ingredients of the

Rankin-Selberg integral were theta series and non-holomorphic Eisenstein series. Now that the eigenform is vector-valued however, either the theta series or the Eisenstein series must also be vector-valued and we go with the former. To allow the Rankin-Selberg method of Section 2.4 to work in this case we will require the coefficients of this vector-valued theta series to have certain properties, and the existence of such a theta series is contingent on the type of representation ρ we take. This section shows how to obtain the desired vector-valued theta series and we also give examples for when this theta series is a cusp form.

We need ρ to be such that there exists a V-valued pluriharmonic polynomial with respect to which ρ acts as right translation; such polynomials will form the Fourier coefficients of the theta series. This is achieved through the work of Kashiwara and Vergne of [KV78], specific cases of which are now summarised.

Definition 5.2.1. Let $\mathbb{C}[M_n]$ denote the ring of complex polynomials on $n \times n$ matrices, and for each $1 \leq i \leq j \leq n$ denote $\Delta_{i,j} := \sum_{k=1}^{n} \frac{\partial^2}{\partial x_{ik} \partial x_{jk}}$. Then a polynomial $P \in \mathbb{C}[M_n]$ is called pluriharmonic if

$$(\Delta_{i,j} P)(x) = 0 \qquad \text{for all } 1 \leq i \leq j \leq n.$$

The space of pluriharmonic polynomials is denoted by $\mathcal{P}_{\mathcal{H}}$.

If V is a finite-dimensional complex vector space and $P(x)$ is a V-valued polynomial on M_n, then it can be written $P = (P_1, \ldots, P_r)$, where $P_i \in \mathbb{C}[M_n]$ and $r = \dim(V)$. In this case we say that P is pluriharmonic if each P_i is pluriharmonic.

Let $O_n(\mathbb{C}) = \{\alpha \in GL_n(\mathbb{C}) \mid \alpha^T \alpha = I_n\}$ be the orthogonal group. Then the group $O_n(\mathbb{C}) \times GL_n(\mathbb{C})$ acts on $\mathbb{C}[M_n]$ by

$$((g, h) \cdot P)(x) = P(g^{-1} x h),$$

and this preserves the subspace $\mathcal{P}_{\mathcal{H}}$.

If (φ, V_φ) is an irreducible representation of $O_n(\mathbb{C})$ then let $\mathcal{P}_{\mathcal{H}}(\varphi)$ denote the space of all V_φ-valued pluriharmonic polynomials $P(x)$ such that $\varphi(g)^{-1} P(x) = P(gx)$. Let Σ denote the set of irreducible $\varphi \in \widehat{O_n(\mathbb{C})}$ such that $\mathcal{P}_{\mathcal{H}}(\varphi) \neq 0$, and for each such φ let $\tau(\varphi)$ denote the representation of $GL_n(\mathbb{C})$ satisfying $(\tau(\varphi)(g^T)P)(x) = P(xg)$. Kashiwara and Vergne show, in [KV78], that $\tau(\varphi)$ is irreducible. It is such pluriharmonic polynomials that form the coefficients for the vector-valued theta series. Therefore the ρ we consider are restricted by the set Σ and the representations $\tau(\varphi)$. Such representations have been determined and described explicitly in [KV78] and, after some brief representation theory, is summarised below.

To set the scene for the description of these representations, let us recap some general basic representation theory and the notion of the highest weight. Given a representation (ρ, V) of a complex matrix Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$, then a *weight* is a linear functional $\mu : \mathfrak{h} \to \mathbb{C}$ such that there exists a $0 \neq v \in V$ upon which $\rho|_{\mathfrak{h}}$ acts as the scalar

μ – i.e. $\rho(H)v = \mu(H)v$ for all $H \in \mathfrak{h}$. One can define a partial ordering on the weights of ρ, and the *highest weight* is a weight that is maximal with respect to this ordering. These weights are important since every finite-dimensional irreducible representation has one, and so such representations can be parameterised by their highest weight. This is how the representations of $O_n(\mathbb{C})$ and $GL_n(\mathbb{C})$ are treated in the following description of [KV78]. If $\ell = \dim_{\mathbb{C}}(\mathfrak{h})$ then the ℓ-tuple $(m_1, \ldots, m_\ell)$ is notation that corresponds to the highest weight $\mu(H) = \lambda_{H,1}^{m_1} \cdots \lambda_{H,\ell}^{m_\ell}$, where $\lambda_{H,i}$ are the eigenvalues of the matrix H.

If n is odd. Say $n = 2\ell + 1$, then we can parameterise the irreducible representations of $O_n(\mathbb{C})$ as follows. Let $m_1 \geq m_2 \geq \cdots \geq m_\ell \geq 0$ where $m_i \in \mathbb{Z}$. Then $(m_1, \ldots, m_\ell)$ is the highest weight of a representation of $SO_n(\mathbb{C})$ and we have $O_n(\mathbb{C}) \cong SO_n(\mathbb{C}) \times \mathbb{Z}/2\mathbb{Z}$. Hence the irreducible representations of $O_n(\mathbb{C})$ are parameterised by the ℓ+1-tuple $(m_1, \ldots, m_\ell; \epsilon)$, which notation corresponds to $(m_1, \ldots, m_\ell) \otimes \epsilon$, where $\epsilon = \pm 1$ and the m_j are as above.

The finite-dimensional irreducible representations of $GL_n(\mathbb{C})$ are parameterised by n-tuples $(m_1, \ldots, m_n)$, with $m_1 \geq m_2 \geq \cdots \geq m_n$ and each $m_j \in \mathbb{Z}$.

Theorem 5.2.2 ([KV78], p. 25)**.** *If $n = 2\ell + 1$ is odd, then every $\varphi \in \widehat{O_n(\mathbb{C})}$ belongs to Σ. If $\varphi = (m_1, \ldots, m_\ell; \epsilon)$ and $\epsilon = (-1)^{\sum_j m_j}$, then*

$$\tau(\varphi) = (m_1, \ldots, m_\ell, \mathbf{0}_{n-\ell}).$$

On the other hand, if $\epsilon = (-1)^{1+\sum_j m_j}$ and $0 \leq r \leq \ell$ is an integer such that $m_r \neq 0$ and $\varphi = (m_1, \ldots, m_r, \mathbf{0}_{n-r}; \epsilon)$, then

$$\tau(\varphi) = (m_1, \ldots, m_r, \mathbf{1}_{n-2r}, \mathbf{0}_r).$$

If n is even. Say $n = 2\ell$, then we parameterise the elements of Σ in a different way. For $x, y \in M_{n\times\ell}(\mathbb{C})$ define

$$\Delta_j(x) := \det \begin{pmatrix} x_{11} & \cdots & x_{1j} \\ \vdots & \ddots & \vdots \\ x_{j1} & \cdots & x_{jj} \end{pmatrix}, \qquad 1 \leq j \leq \ell, n,$$

$$\widetilde{\Delta}_j(x, y) := \det \begin{pmatrix} \mathbf{x}_1^T & y_{1,j+1} & \cdots & y_{1\ell} \\ \mathbf{x}_2^T & y_{2,j+1} & \cdots & y_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_{2\ell-j}^T & y_{2\ell-j,j+1} & \cdots & y_{2\ell-j,\ell} \end{pmatrix}, \qquad 0 \leq j \leq n,\ 2\ell - j \leq n,$$

where $\mathbf{x}_i \in \mathbb{C}^\ell$ is the ith column of x. With $m_1 \geq m_2 \geq \cdots \geq m_\ell \geq 0$, all integers as before, let $(m_1, \ldots, m_\ell)_+$ denote the irreducible representation of $O_n(\mathbb{C})$ generated by the products $\Delta_1(x)^{m_1-m_2} \cdots \Delta_{\ell-1}^{m_{\ell-1}-m_\ell} \Delta_\ell^{m_\ell}(x)$ under left translation. On the other hand, for any $1 \leq r \leq \ell$ such that $m_r \neq 0$ and $m_{r+1} = 0$, let $(m_1, \ldots, m_\ell)_-$ denote the irreducible representation of $O_n(\mathbb{C})$ generated by $\Delta_1(x)^{m_1-m_2} \cdots \Delta_{r-1}(x)^{m_{r-1}-m_r} \Delta_r(x)^{m_r-1} \widetilde{\Delta}_r(x, y)$ under left translation. Note that $\widetilde{\Delta}_\ell(x, y) = \Delta_\ell(x)$, and therefore $(m_1, \ldots, m_\ell)_+ = (m_1, \ldots, m_\ell)_-$, if $m_\ell \neq 0$.

Theorem 5.2.3 ([KV78], p. 27)**.** *If $n = 2\ell$ is even then we have $\Sigma = \Sigma_+ \cup \Sigma_-$, where $\Sigma_\pm := \{(m_1, \ldots, m_\ell)_\pm\}$. If $\varphi = (m_1, \ldots, m_\ell)_+ \in \Sigma_+$, then*

$$\tau(\varphi) = (m_1, \ldots, m_\ell, \mathbf{0}_{n-\ell}),$$

whereas if $\varphi = (m_1, \ldots, m_r, \mathbf{0}_{\ell-r})_- \in \Sigma_-$ with $m_r \neq 0$, then

$$\tau(\varphi) = (m_1, \ldots, m_r, \mathbf{1}_{n-2r}, \mathbf{0}_r).$$

For any $\rho \in \tau(\Sigma) := \{\tau(\varphi) \mid \varphi \in \Sigma\}$ we can now construct a vector-valued theta series of weight $\rho_{\frac{n}{2}}$ with pluriharmonic coefficients. If $\tau \in S_+$ is fixed and $\lambda \in \mathcal{S}(M_n(\mathbb{Q}_\mathbf{f}))$ is a Schwartz-Bruhat function, as defined at the start of Section 4.4, then the Jacobi theta series for variables $z \in \mathbb{H}_n$ and $u \in M_n(\mathbb{C})$ is given, as in [Shi00, p. 262], by

$$\begin{aligned} g(u, z; \lambda) &:= \sum_{\xi \in M_n(\mathbb{Q})} \lambda(\xi_\mathbf{f}) e\left(U^T (I_n \otimes 4iy)^{-1} U + \mathrm{tr}(\xi^T \tau \xi z) + \mathrm{tr}(u \sqrt{2\tau}\xi)\right), \\ U &:= (u_{11}, \ldots, u_{n1}, \ldots, u_{1n}, u_{nn}) \in \mathbb{C}^{n^2}, \end{aligned} \tag{5.2.1}$$

where $\sqrt{\tau}$ is a matrix such that $(\sqrt{\tau})^2 = \tau$. Properties of this theta series are detailed in Theorem A3.3 of [Shi00], most notably the action $\lambda \mapsto {}^\alpha\lambda$ on Schwartz-Bruhat functions – previously discussed in Section 4.4 – and the transformation formula

$$\mathfrak{J}(\alpha, z) g(\alpha \cdot (u, z); {}^\alpha\lambda) = g(u, z; \lambda), \tag{5.2.2}$$

in which

$$\begin{aligned} \mathfrak{J}(\alpha, z) &:= \begin{cases} j(\alpha, z)^{\frac{n}{2}} & \text{if } n \text{ is even,} \\ h(\alpha, z)^n & \text{if } n \text{ is odd,} \end{cases} \\ \alpha \cdot (u, z) &:= \left(\widetilde{\mu(\alpha, z)} u, \alpha \cdot z\right), \end{aligned}$$

and which holds (the transformation) for all $\alpha \in G$ if n is even and all $\alpha \in \mathfrak{M}$ if n is odd.

Let (ρ, V) be a representation of $GL_n(\mathbb{C})$ with $\rho \in \tau(\Sigma)$, and let $r := \dim_\mathbb{C}(V)$. By definition of $\tau(\Sigma)$ there exists a V-valued pluriharmonic polynomial $P = (P_1, \ldots, P_r)$, with each $P_i \in \mathbb{C}[M_n]$, such that $\rho(g)P(x) = P(xg^T)$. By Remark (6.5) of [KV78] we can and do select $P(x)$ to be a highest-weight vector with respect to ρ. Then we define the following V-valued theta series:

$$\theta(z, \lambda; P) := \sum_{\xi \in M_n(\mathbb{Q})} \lambda(\xi_\mathbf{f}) P(\sqrt{2\tau}\xi) e\left(\mathrm{tr}(\xi^T \tau \xi z)\right).$$

Theorem 5.2.4. *Let $\alpha \in G$ or $G \cap \mathfrak{M}$ according to whether n is even or odd, and define the factor of automorphy $\mathfrak{J}_\rho(\alpha, z) := \mathfrak{J}(\alpha, z)\rho(\mu(\alpha, z))$. We have*

$$\theta(\alpha z, {}^\alpha\lambda; P) = \mathfrak{J}_\rho(\alpha, z)\theta(z, \lambda; P).$$

Proof. Let $\partial := \left(\frac{\partial}{\partial u_{ij}}\right)_{i,j}$ be a differential operator on functions $M_n(\mathbb{C}) \to \mathbb{C}$. Let P be a

V-valued pluriharmonic polynomial of the form $P = (P_1, \ldots, P_m)$ (with respect to a fixed basis of V) and define the differential operator $P(\partial)$. This operates on $f : M_n(\mathbb{C}) \to \mathbb{C}$ component-wise as $P(\partial)f := (P_1(\partial)f, \ldots, P_m(\partial)f)$, and this defines a V-valued function on $M_n(\mathbb{C})$.

Since $P(\partial)e(\mathrm{tr}(ua)) = 2\pi i P(a)e(\mathrm{tr}(ua))$ for any $a \in M_n(\mathbb{C})$, it is clear that

$$2\pi i\theta(z, \lambda; P) = (P(\partial)g(u, z; \lambda))\big|_{u=0}.$$

By Lemma A3.6 of [Shi00] we also have

$$\begin{aligned}
&P(\partial)e\left(U^T(I_n \otimes 4iy)^{-1}U + \mathrm{tr}(\xi^T\tau\xi z) + \mathrm{tr}(u\sqrt{2\tau}\xi)\right)|_{u=0} \\
&= P(\partial)e\left(\mathrm{tr}(\xi^T\tau\xi z) + \mathrm{tr}(u\sqrt{2\tau}\xi)\right)|_{u=0}.
\end{aligned}$$

Applying $P(\partial)|_{u=0}$ to both sides of the transformation formula of (5.2.2), along with the observation that $P\left(\sqrt{2\tau}\xi\widetilde{\mu(\alpha, z)}\right) = \rho(\mu(\alpha, z))^{-1}P(\sqrt{2\tau}\xi)$, then gives the theorem. □

Both $\theta(z, \lambda; P)$ above and the scalar theta series $\theta(z, \lambda)$ of (4.4.1) are obtained from the Jacobi theta series $g(u, z; \lambda)$ of (5.2.1) by the application of a differential operator, and so they share a lot of properties. Both Proposition A3.17, concerning the Fourier expansion of $\theta(z, \lambda)$, and Proposition A3.19, concerning the level of $\theta(z, \lambda)$, of [Shi00] hold also for $\theta(z, \lambda; P)$ when one replaces, in those propositions, $\det^\mu$ with $P(\sqrt{2\tau}\xi)$.

Let χ be a Hecke character of conductor $\mathfrak{f}$ and fix an element $Q \in GL_n(\mathbb{Q}_\mathfrak{f})$. The theta series that plays a role in the Rankin-Selberg method is defined by

$$\begin{aligned}
\theta_{\rho,\chi}(z) &:= \theta(z, \lambda; P), \\
\lambda(x) &:= \prod_p \chi_p(|Q|)\lambda'_p(Q^{-1}x), \\
\lambda'_p(y) &:= \begin{cases} 1 & \text{if } y \in M_n(\mathbb{Z}_p) \text{ and } p \nmid \mathfrak{f}, \\ \chi_p(|y|) & \text{if } y \in GL_n(\mathbb{Z}_p) \text{ and } p \mid \mathfrak{f}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Through analogy to Proposition A3.19 of [Shi00], we have that

$$\theta_{\rho,\chi}(z) = \sum_{x \in M_n(\mathbb{Z})} (\chi_\infty\chi^*)^{-1}(|x|)P(\sqrt{2\tau}x)e_\infty(\mathrm{tr}(x^T\tau xz))$$

belongs to $\mathcal{M}_{\rho_{n/2}}(\Gamma[(\mathfrak{b}')^{-1}, \mathfrak{b}'\mathfrak{c}'], \chi^{-1}\epsilon_\tau)$, where ϵ_τ (previously ρ_τ) is the quadratic character of conductor $\mathfrak{h}$ corresponding to the extension $\mathbb{Q}(i^{[\frac{n}{2}]}|2\tau|^{\frac{1}{2}})/\mathbb{Q}$ and the ideals $\mathfrak{b}'$ and $\mathfrak{c}'$ are determined as follows. Let $\mathfrak{r}$ be the ideal such that $g^T 2\tau g \in \mathfrak{r}$ for all $g \in Q\mathbb{Z}^n$, let $\mathfrak{a} := \mathfrak{r}^{-1} \cap \mathbb{Z}$, and let $\mathfrak{t}$ be the ideal such that $h^T(2\tau)^{-1}h \in 4\mathfrak{t}^{-1}$ for all $h \in \widetilde{Q}\mathbb{Z}^n$. We can take

$$(\mathfrak{b}', \mathfrak{c}') = \begin{cases} (2^{-1}\mathfrak{r}, \mathfrak{h} \cap \mathfrak{f} \cap \mathfrak{r}^{-1}\mathfrak{f}^2\mathfrak{t}) & \text{if } n \in 2\mathbb{Z}, \\ (2^{-1}\mathfrak{a}^{-1}, \mathfrak{h} \cap \mathfrak{f} \cap 4\mathfrak{a} \cap \mathfrak{a}\mathfrak{f}^2\mathfrak{t}) & \text{if } n \notin 2\mathbb{Z}. \end{cases} \tag{5.2.3}$$

This section is now completed by a result on the cuspidality of this theta series. We say that χ is *odd* if $\chi_{\mathfrak{f}}(-1) = -1$ (equivalently $\chi_\infty(-1) = -1$) and we show that for odd χ one can choose τ and Q in order to guarantee cuspidality of $\theta_{\rho,\chi}$.

If $\iota = \left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right)$, $\mathrm{pr}(\sigma) = \left(\begin{smallmatrix} I_n & b_\sigma \\ 0 & I_n \end{smallmatrix}\right)$, $x \in M_n(\mathbb{Q})$, and $\tau \in S_+$ is fixed, then Theorem A3.3 (5) and Equation (A3.3) of [Shi00] tell us that

$$({}^{\sigma}\lambda)(x) = e_{\mathbf{f}}(\mathrm{tr}(x^T \tau x b_\sigma^T))\lambda(x), \tag{5.2.4}$$

$$({}^{\iota}\lambda)(x) = i^d |2\tau|^{-\frac{n}{2}} \int_{M_n(\mathbb{Q})_{\mathbf{f}}} \lambda(y) e_{\mathbf{f}}(-\,\mathrm{tr}(x^T 2\tau y)) d_{\mathbf{f}} y, \tag{5.2.5}$$

where d is defined by (4.1.2), the Haar measure $d_{\mathbf{f}}y = \prod_p d_p y$ of $M_n(\mathbb{Q}_{\mathbf{f}})$, and recall each $d_p y$ from Section 4.4 as the Haar measure on $M_n(\mathbb{Q}_p)$ such that $\mathrm{Vol}(Y M_n(\mathbb{Z}_p)) = |\det(Y)|_p^{n/2}$ for any $Y \in M_n(\mathbb{Q}_p)$.

For any $X \in M_n(\mathbb{Z})$ and $s \in S_+$, define the generalised quadratic Gauss sum by

$$G_n'(\chi, X, s, \mathfrak{f}) := \sum_{a \in M_n(\mathbb{Z}/N(\mathfrak{f})\mathbb{Z})} \chi_{\mathfrak{f}}^{-1}(|a|) e\left(\frac{\mathrm{tr}(X^T a - \tau[Q] a s a^T)}{N(\mathfrak{f})}\right),$$

where, for simplicity, we have put $\tau[Q] := Q^T \tau Q$. By Theorem A3.3 (2) of [Shi00] we have that ${}^{\iota\sigma}\lambda = {}^{\iota}({}^{\sigma}\lambda)$ and this integral is calculated as follows.

Lemma 5.2.5. *Let χ be a Hecke character of conductor $\mathfrak{f}$ and put $F_p := \mathrm{ord}_p(\mathfrak{f})$. Assume that τ and σ are as above, that $b = b_\sigma \in S(\mathbb{Z}_p)$, and take a $Q \in GL_n(\mathbb{Q}_{\mathbf{f}})$ such that $Q \in p^{-F_p} M_n(\mathbb{Z}_p)$ for any prime $p \mid \mathfrak{f}$. Then the value of ${}^{\iota}({}^{\sigma}\lambda)(x)$ is non-zero if and only if*

$$\tau z b z^T - 2x^T \tau z \in M_n(\mathbb{Z}_p),$$

for all primes p and for all

$$z \in \begin{cases} M_n(\mathbb{Z}_p) & \text{if } p \mid \mathfrak{f}, \\ Q M_n(\mathbb{Z}_p) & \text{if } p \nmid \mathfrak{f}. \end{cases}$$

For such an x we have

$${}^{\iota}({}^{\sigma}\lambda)(x) = i^d \bar{\chi}(|Q|)(QN(\mathfrak{f}))^{-\frac{n^2}{2}} |2\tau|^{-\frac{n}{2}} G_n'(\bar{\chi}, 2N(\mathfrak{f}) Q^T \tau x, N(\mathfrak{f})^{-1} b, \mathfrak{f}). \tag{5.2.6}$$

Proof. Write $i^{-d}({}^{\iota}({}^{\sigma}\lambda)) = \prod_p {}^{\iota}({}^{\sigma}\lambda)_p$ and first consider the local integrals for $p \mid \mathfrak{f}$; in this case we have from (5.2.4) and (5.2.5) that ${}^{\iota}({}^{\sigma}\lambda)_p(x)$ is equal to

$$\begin{aligned} &|\det(2\tau)|_p^{\frac{n}{2}} \chi_p(|Q|) \int_{QGL_n(\mathbb{Z}_p)} \chi_p(|Q^{-1}y|) e_p\left(\mathrm{tr}(\tau y b y^T - 2x^T \tau y)\right) d_p y \\ &\qquad = \chi_p(|Q|) |\det(Q 2\tau)|_p^{\frac{n}{2}} \int_{GL_n(\mathbb{Z}_p)} \chi_p(|y|) e_p\left(\mathrm{tr}(\tau Q y b (Qy)^T - 2x^T \tau Q y)\right) d_p y. \end{aligned}$$

Since the local conductor of χ at p is p^{F_p}, this becomes

$$\begin{aligned} &\chi_p(|Q|) |\det(Q 2\tau)|_p^{\frac{n}{2}} \sum_{a \in M_n(\mathbb{Z}/p^{F_p}\mathbb{Z})} \chi_p(|a|) e\left(\mathrm{tr}(2x^T \tau Q a - \tau[Q] a b a^T)\right) \\ &\qquad\qquad \times \int_{p^{F_p} M_n(\mathbb{Z}_p)} e_p\left(\mathrm{tr}(\tau Q y b (Qy)^T - 2x^T \tau Q y)\right) d_p y. \end{aligned}$$

The integral on the second line is non-zero if and only if the integrand is a constant function in y – i.e. if and only if $\tau z b z^T - 2x^T\tau z \in M_n(\mathbb{Z}_p)$ for any $z \in M_n(\mathbb{Z}_p)$– at which point the integral is $p^{-F_p(n^2/2)}$. This gives the non-zero conditions for when $p \mid \mathfrak{f}$.

When $p \nmid \mathfrak{f}$ the local integral $({}^{\iota\sigma}\lambda)_p(x)$ is equal to

$$|\det(2\tau)|_p^{\frac{n}{2}} \int_{QM_n(\mathbb{Z}_p)} e_p\left(\mathrm{tr}(\tau y b y^T - 2x^T\tau y)\right) d_p y$$

and this is non-zero if and only if we have the non-zero condition given in this lemma at which point, by definition of the measure, it is $|\det(Q)|_p^{\frac{n}{2}}$.

Multiplying the local integrals together for all p then gives the last identity of the lemma – (5.2.6). □

Proposition 5.2.6. *If* $\det(X) = 0$, *p is an odd prime,* $\tau = \mathrm{diag}[\tau_1, \ldots, \tau_n]$ *is diagonal,* $Q \in GL_n(\mathbb{Q}_{\mathbf{f}})$ *is upper triangular with* $q_{1n} = \cdots = q_{n-1,n} = 0$, *and* χ *is odd of conductor* p, *then*

$$G'_n(\chi, X, s, p) = 0.$$

Proof. In the base $n = 1$ case, $0 = X \in \mathbb{Z}$ and we can write

$$G'_1(\chi, X, s, p) = \sum_{n\in\mathbb{F}_p^\times} \chi_p^{-1}(n) e^{-2\pi i\tau Q^2 \frac{sn^2}{p}} = \sum_{n=1}^{\frac{p-1}{2}} [\chi_p^{-1}(n) + \chi_p^{-1}(-n)] e^{-2\pi i\tau Q^2 \frac{sn^2}{p}},$$

for $Q \in \mathbb{Q}$ and $\tau, s \in \mathbb{Z}$; this is zero if χ is odd.

For the general n case, first let M_j^n be the $(n-1)\times(n-1)$ matrix obtained from any $n \times n$ matrix M by removing the jth row and the nth column. By a change of basis followed by a change of variables in a we can assume that

$$X = \begin{pmatrix} X_n^n & \mathbf{0} \\ \mathbf{x} & 0 \end{pmatrix},$$

where $\begin{pmatrix} \mathbf{x} & 0 \end{pmatrix} \in \mathbb{Z}^n$ is the nth row of X. Let $\mathbf{a}_i$ be the ith column of a, and let $\boldsymbol{\alpha}_i$ be the ith column of aQ^T. Then

$$\mathrm{tr}(X^T a) = \mathrm{tr}((X_n^n)^T a_n^n) + \sum_{i=1}^{n-1} x_{ni} a_{ni},$$

$$\mathrm{tr}(\tau[Q]a^T s a) = \mathrm{tr}(\tau (aQ^T)^T s (aQ^T)) = \sum_{i=1}^{n} \tau_i \boldsymbol{\alpha}_i^T s \boldsymbol{\alpha}_i,$$

and so within the sum defining $G'_n(\chi, X, s, p)$ appears the following subsum

$$\sum_{\mathbf{a}_n\in\mathbb{F}_p^n} \chi_p^{-1}(|a|) e\left(-\frac{\tau_n q_n^2 \mathbf{a}_n^T s \mathbf{a}_n}{p}\right). \tag{5.2.7}$$

We have been able to separate the variables as such by the specific form of Q and by using $\boldsymbol{\alpha}_n = q_n \mathbf{a}_n$. The proof is completed by showing that the sum in (5.2.7) above is zero if χ is odd. By Lemma A1.5 of [Shi00], there exists $u \in GL_n(\mathbb{Z})$ such that $s' := \tilde{u}su^{-1} = \mathrm{diag}[s_1, \ldots, s_n]$ is diagonal. Using the expansion

$$|a| = \sum_{j=1}^{n} (-1)^{n+j} a_{jn} |a_j^n|$$

the sum of (5.2.7) can be written as

$$\mathrm{sgn}(|u|) \sum_{\mathbf{a}_n \in \mathbb{F}_p^n} \chi_p^{-1}(|ua|) e\left(-\tfrac{\tau_n q_n^2 (u\mathbf{a})_n^T s' (u\mathbf{a})_n}{p}\right),$$

the absolute value of which, after a change of variables, becomes

$$\sum_{(a_{1n},\ldots,a_{nn}) \in \mathbb{F}_p^n} \chi_p^{-1}\left(\sum_{j=1}^{n} (-1)^{n+j} a_{jn} |(ua)_j^n|\right) e\left(-p^{-1}\tau_n q_n^2 \sum_{j=1}^{n} s_j a_{jn}^2\right). \tag{5.2.8}$$

In the base $n = 1$ case, (5.2.8) above becomes $G_1'(\chi, X, s, p)$ which we have shown to be zero at the beginning of this proof. So now assume that the $n-1$ degree sum corresponding to (5.2.8) is zero. If one of the $a_{jn} = 0$ in (5.2.7), then it becomes the $n-1$ degree sum and is therefore zero. So we can assume by induction that $(a_{1n}, \ldots, a_{nn}) \in (\mathbb{F}_p^\times)^n$, which set can be partitioned as

$$(\mathbb{F}_p^\times)^n = \bigsqcup_{i_1,\ldots,i_n=0}^{1} (-1)^{i_j} (\mathbb{F}_p^-)^n,$$

where $\mathbb{F}_p^- := \{1, \ldots, \frac{p-1}{2}\}$. This partition can be seen by writing any $(t_1, \ldots, t_n) \in (\mathbb{F}_p^\times)^n$ as $((-1)^{i_1} t_1', \ldots, (-1)^{i_n} t_n')$ where $t_j' = |t_j''|$ and t_j'' is the representative of t_j taken in $\{\pm 1, \ldots, \pm\frac{p-1}{2}\}$. The aim is to re-write the sum of (5.2.8) over $(\mathbb{F}_p^-)^n$. To this end, notice that as $(a_{1n}, \ldots, a_{nn}) \mapsto ((-1)^{i_1} a_{1n}', \ldots, (-1)^{i_n} a_{nn}')$ we have

$$\begin{aligned}
(5.2.8) \mapsto &\sum_{(a_{1n}',\ldots,a_{nn}') \in (\mathbb{F}_p^-)^n} \sum_{\mathbf{i} \in \mathbb{F}_2^n} \chi_p^{-1}(|a|_{\mathbf{i}}) e\left(-p^{-1}\tau_n q_n^2 \sum_{j=1}^{n} s_j (a_{jn}')^2\right), \\
|a|_{\mathbf{i}} : = &\sum_{j=1}^{n} (-1)^{n+j+i_j} a_{jn}' |(ua)_j^n|,
\end{aligned}$$

where $\mathbf{i} = (i_1, \ldots, i_n) \in \mathbb{F}_2^n$. The argument of the exponential is unchanged by the transformation $((-1)^{i_1} a_{1n}', \ldots, (-1)^{i_n} a_{nn}') \mapsto -((-1)^{i_1} a_{1n}', \ldots, (-1)^{i_n} a_{nn}')$, yet in the coefficients we see $|a|_{\mathbf{i}} \mapsto -|a|_{\mathbf{i}}$. Hence we can pair up the coefficients of the exponential as follows. Let $\sim$ be an equivalence relation on $\mathbb{F}_2^n$ defined by $\mathbf{i}_1 \sim \mathbf{i}_2$ if and only if $\mathbf{i}_1 = \mathbf{i}_2 + \mathbf{1}$. Then (5.2.8) becomes

$$\sum_{(a_{n1}',\ldots,a_{nn}') \in (\mathbb{F}_p^-)^n} \sum_{\mathbf{i} \in \mathbb{F}_2^n/\sim} \left[\chi_p^{-1}(|a|_{\mathbf{i}}) + \chi_p^{-1}(-|a|_{\mathbf{i}})\right] e\left(-p^{-1}\tau_n q_n^2 \sum_{j=1}^{n} s_j (a_{jn}')^2\right)$$

which is zero, since χ is odd. □

Theorem 5.2.7. *Let χ be an odd non-trivial character of square-free conductor prime to 2. Then there are choices of $\tau \in S_+$ and $Q \in GL_n(\mathbb{Q}_{\mathbf{f}})$ such that $\theta_{\rho,\chi}$ is a cusp form.*

Proof. Let $\theta(z,\lambda;P)$ be the corresponding theta series to $\theta_{\rho,\chi}$. Let $\Gamma' := \Gamma[(\mathfrak{b}')^{-1}, \mathfrak{b}'\mathfrak{c}']$ be the congruence subgroup of the theta series given in (5.2.3). At the beginning of Section 2.3.2 we defined the group G', parabolic subgroups $P^{n,r}$ for any integer $0 \le r \le n$, and respective metaplectic lifts $\mathfrak{G}$ and $\mathfrak{P}^{n,r}$. Let $G'(R) = \{x \in G' \mid x_\infty \in Sp_n(R)\}$, $\mathfrak{G}(R) = \mathrm{pr}^{-1}(G'(R))$, $P^{n,r}(R) = P^{n,r} \cap M_{2n}(R)$, and $\mathfrak{P}^{n,r}(R) = \mathrm{pr}^{-1}(P^{n,r}(R))$, for any subring R of $\mathbb{Q}$. Let

$$X := \begin{cases} \Gamma' \backslash G'(\mathbb{Z})/P^{n,n-1}(\mathbb{Z}) & \text{if } n \text{ is even,} \\ \Gamma' \backslash \mathfrak{G}'(\mathbb{Z})/\mathfrak{P}^{n,n-1}(\mathbb{Z}) & \text{if } n \text{ is odd.} \end{cases}$$

These represent the $n-1$-dimensional cusps. To show that the theta series is a cusp form, it is enough to show that

$$\Phi\left(\theta(z,\lambda;P)|_{\rho_{\frac{n}{2}}}\alpha\right) = 0,$$

where Φ is the Siegel Phi operator of Section 2.3.1 and α runs over a set of representatives of X.

The plan is to find these representatives explicitly. First assume that $\Gamma' = \Gamma[p,p]$, where p is a prime; the kernel of the surjective projection map $Sp_n(\mathbb{Z}) \to Sp_n(\mathbb{F}_p)$ belongs to $\Gamma[p,p]$. So we can just find representatives of

$$C := \begin{cases} Q(\mathbb{F}_p) \backslash G'(\mathbb{F}_p)/P^{n,n-1}(\mathbb{F}_p) & \text{if } n \text{ is even,} \\ Q(\mathbb{F}_p) \backslash \mathfrak{G}'(\mathbb{F}_p)/\mathfrak{P}^{n,n-1}(\mathbb{F}_p) & \text{if } n \text{ is odd,} \end{cases}$$

where $Q(\mathbb{F}_p) := \{\mathrm{diag}[a,\tilde{a}] \mid a \in GL_n(\mathbb{F}_p)\}$, and lift back. The Bruhat decomposition gives us

$$Sp_n(\mathbb{F}_p) = P(\mathbb{F}_p)P^{n,n-1}(\mathbb{F}_p) \cup P(\mathbb{F}_p)\iota P^{n,n-1}(\mathbb{F}_p).$$

Now note that $P(\mathbb{F}_p) = Q(\mathbb{F}_p)R(\mathbb{F}_p)$, where $R(\mathbb{F}_p)$ is the set of all matrices $r(s) := \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right)$ with $s^T = s \in M_n(\mathbb{F}_p)$. Hence a subset of $\{r(s), r(s)\iota \mid s^T = s \in M_n(\mathbb{F}_p)\}$ provides a set of representatives for C. Lifting these matrices back to $Sp_n(\mathbb{Z})$ gives the desired set of representatives.

In general $\Gamma' = \Gamma[m,m]$, where $m = p_1 \cdots p_\ell$ is square-free, and one can use the Chinese remainder theorem to show that $Sp_n(\mathbb{Z}/m\mathbb{Z}) = Sp_n(\mathbb{F}_{p_1}) \times \cdots \times Sp_n(\mathbb{F}_{p_\ell})$. So this reduces everything down to the previous case of a single prime.

If χ is now an odd character of conductor p then we can choose τ and Q that satisfy the conditions of Lemma 5.2.5 and Proposition 5.2.6 and are such that $(\mathfrak{b}', \mathfrak{c}') = ((2p)^{-1}, 4p^2)$ (for example, taking $\tau = 2pI_n$ and $Q = (2p)^{-1}I_n$ in (5.2.3)); for such ideals $\Gamma' = \Gamma[2p,2p]$. The rest of the proof is completed by showing that, for such a choice, $\theta_{\rho,\chi}(z) = \theta(z,\lambda;P)$ is a cusp form. For any $\alpha \in Sp_n(\mathbb{Q})$ if n is even, or $\alpha \in \mathfrak{M}$ if n is odd, we have that

$$\theta(z,\lambda;P)|_{\rho_{\frac{n}{2}}}\alpha = \theta(z, {}^{\alpha^{-1}}\lambda; P),$$

and so we show that if $|x| = 0$ then $({}^{\alpha^{-1}}\lambda)(x) = 0$ for all $\alpha \in X$. We have already shown that, after projection to $Sp_n(\mathbb{F}_2) \times Sp_n(\mathbb{F}_p)$, these representatives are distinguishable as

$$(r(s), r(s')), \tag{5.2.9}$$
$$(r(s), r(s')\iota), \tag{5.2.10}$$
$$(r(s)\iota, r(s')), \tag{5.2.11}$$
$$(r(s)\iota, r(s')\iota). \tag{5.2.12}$$

It is enough to check the support of the local Schwartz-Bruhat functions λ_2 and λ_p after transformation by the above elements. For the cusp of (5.2.9), that we have $({}^{\alpha^{-1}}\lambda)_v(x) = 0$ for $|x| = 0$ and $v \in \{2, p\}$ is immediate from the action of parabolic elements, see (5.2.4). For the cusp of (5.2.11) note that χ_2 is trivial, so that ${}^{\iota m(-s)}\lambda_2$ is just the local version of (5.2.4) at 2 and this is zero when $|x| = 0$. For the cusps of (5.2.10) and (5.2.12) we obtain ${}^{\iota m(-s')}\lambda_p$ at the pth place and, by Lemma 5.2.5 and Proposition 5.2.6, this is zero if $|x| = 0$. □

5.3 The Rankin-Selberg integral expression

Notation. $k \in \frac{1}{2}\mathbb{Z}$ – integral or half-integral weight.
$[k] = k$ if $k \in \mathbb{Z}$; $[k] = k - \frac{1}{2}$ if $k \notin \mathbb{Z}$.
V – finite-dimensional complex vector space.
$\rho : GL_n(\mathbb{C}) \to GL(V)$ – rational representation; $\rho_k = h^{k-[k]} \otimes \det^{[k]} \otimes \rho$.
$\Delta(z) = |\,\mathfrak{Im}(z)|$ for $z \in \mathbb{H}_n$.
$\mu(\gamma, z) = c_\gamma z + d_\gamma$ if $\gamma \in G$ and $z \in \mathbb{H}_n$.
$j_\gamma^k(z) = |\mu(\gamma, z)|$ if $k \in \mathbb{Z}$ and $\gamma \in G$;
$j_\gamma^k(z) = h_\gamma(z)|\mu(\gamma, z)|^{[k]}$ if $k \notin \mathbb{Z}$ and $\gamma \in G \cap \mathfrak{M}$.
$S_+ = \{\xi \in M_n(\mathbb{Q}) \mid \xi^T = \xi, \xi \geq 0\}$.
$P(x)$ – V-valued pluriharmonic polynomial in $\mathbb{C}[M_n]$ which is a highest-weight vector for ρ and coefficient for $\theta_{\rho,\chi}$.
$X = \{x \in M_n(\mathbb{R}) \mid x^T = x, -\frac{1}{2} \leq x_{ij} \leq \frac{1}{2}\}$;
$Y = \{y \in M_n(\mathbb{R}) \mid y^T = y, y > 0\}$.
$\mathsf{X}_p = M_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p)$; $\mathsf{O}_p = GL_n(\mathbb{Z}_p)$;
$\mathsf{X} = GL_n(\mathbb{Q})_\mathbf{f} \cap \prod_p \mathsf{X}_p$; $\mathsf{O} = \prod_p GL_n(\mathbb{Z}_p)$.

This whole section plays out, structurally, in a similar way to Section 2.4 of the scalar case and, methodically, with some key differences. Recall that the integral expression of Section 2.4 was obtained, in three steps, through relations of the Rankin-Selberg integral and the L-function to the auxiliary Dirichlet series $D(s, f, \theta_\chi)$ and $D_\tau(s, f, \chi)$. The main differences lie in extending these series to the vector-valued case, ensuring their desired properties, and accounting for the behaviour of the coefficients of vector-valued theta series. The work of the previous section in ensuring that the coefficients of $\theta_{\rho,\chi}$ are highest-weight vectors with respect to ρ will be of critical use in Section 5.3.2.

5.3.1 Unfolding

For this section take the following set of data: weights $k, \ell \in \frac{1}{2}\mathbb{Z}$; congruence subgroups $\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}]$ and $\Gamma' = \Gamma[(\mathfrak{b}')^{-1}, \mathfrak{b}'\mathfrak{c}']$ such that

$$\begin{aligned} (\mathfrak{b}^{-1}, \mathfrak{bc}) &\subseteq 2\mathbb{Z} \times 2\mathbb{Z} \qquad \text{if } k \notin \mathbb{Z}, \\ ((\mathfrak{b}')^{-1}, \mathfrak{b}'\mathfrak{c}') &\subseteq 2\mathbb{Z} \times 2\mathbb{Z} \qquad \text{if } \ell \notin \mathbb{Z}; \end{aligned}$$

a Hecke character ψ satisfying the usual properties, (2.1.5) and (2.1.6), with $\kappa = k$ and a Hecke character ψ' also satisfying (2.1.5) but with $\mathfrak{c} = \mathfrak{c}'$ and with the further property that $(\psi/\psi')_\infty(x) = \operatorname{sgn}(x_\infty)^{[k-\ell]}$; a non-zero eigenform $f \in \mathcal{S}_{\rho_k}(\Gamma, \psi)$ and $g \in \mathcal{M}_{\rho_\ell}(\Gamma', \psi')$.

By choice of ψ' the Hecke character $\eta = \psi(\psi')^{-1}$ satisfies (2.1.7) with $\kappa = k - \ell$. Fix $\mathfrak{b}$ and $\mathfrak{b}'$, let $\mathfrak{x} := \mathfrak{b} \cap \mathfrak{b}'$, $\mathfrak{y} := \mathfrak{x}^{-1}(\mathfrak{bc} \cap \mathfrak{b}'\mathfrak{c}')$, and put $\Gamma_0 := \Gamma[\mathfrak{x}^{-1}, \mathfrak{xy}]$. The required Eisenstein series remains scalar and is given by

$$E_{k-\ell}(z) := E_{k-\ell}\left(z, \bar{s} + \tfrac{n+1}{2}; \bar{\eta}, \Gamma_0\right),$$

the precise definition of which is given in (2.1.8). Note that $\rho_k(\sqrt{y})$ is a Hermitian operator with respect to the inner product $\prec \cdot, \cdot \succ$. Akin to the first line of the expression of (2.4.6) the integral $\operatorname{Vol}(\Gamma_0 \backslash \mathbb{H}_n)\langle f, gE_{k-\ell}\rangle_{\mathfrak{y}}$ is equal to

$$\begin{aligned} \int_{\Gamma_0\backslash\mathbb{H}_n} \sum_{\gamma \in P\cap\Gamma_0\backslash\Gamma_0} &\prec \rho_k(\mathfrak{Im}(\gamma z))\psi_{\mathfrak{c}}(|a_\gamma|)(f|_{\rho_k}\gamma^{-1})(\gamma z), \psi'_{\mathfrak{c}'}(|a_\gamma|)(g|_{\rho_\ell}\gamma^{-1})(\gamma z) \succ \\ &\times \Delta(\gamma z)^{s+\frac{n+1-k+\ell}{2}} d^\times z, \end{aligned} \tag{5.3.1}$$

where we used the definition, (2.1.8), of the Eisenstein series; that $\psi_p(|a_\gamma|) = 1 = \psi'_q(|a_\gamma|)$ if $p \nmid \mathfrak{c}$ and $q \nmid \mathfrak{c}'$; in order, the following facts

$$j_\gamma^{k-\ell}(z)^{-1} = j^k_{\gamma^{-1}}(\gamma z) j^\ell_{\gamma^{-1}}(\gamma z)^{-1}, \tag{5.3.2}$$

$$\overline{j^k_{\gamma^{-1}}(\gamma z)} = j^k_{\gamma^{-1}}(\gamma z)^{-1}\|\mu(\gamma, z)\|^{-2k}, \tag{5.3.3}$$

$$y = \overline{\mu(\gamma, z)^T}\, \mathfrak{Im}(\gamma z)\mu(\gamma, z), \tag{5.3.4}$$

$$\mu(\gamma, z) = \mu(\gamma^{-1}, \gamma z)^{-1}, \tag{5.3.5}$$

$$\prec \rho(\bar{a}^T)\cdot, \cdot \succ \; = \; \prec \cdot, \rho(a)\cdot \succ, \tag{5.3.6}$$

$$\rho_\kappa(\mu(\gamma^{-1}, \gamma z)^{-1}) = \rho(\mu(\gamma^{-1}, \gamma z)^{-1}) j^\kappa_{\gamma^{-1}}(\gamma z)^{-1} \text{ for } \kappa \in \{k, \ell\}; \tag{5.3.7}$$

and, finally, the definition of the slash operator for vector-valued modular forms. Since f and g are invariant with respect to the slash operator of weights ρ_k and ρ_ℓ respectively and that, by fact in (5.3.4) above, the function $\varphi(z) := \prec \rho_k(y)f(z), g(z) \succ |y|^{s+\frac{n+1-k+\ell}{2}}$ is P-invariant, the application of the unfolding procedure of (2.4.5) to (5.3.1) gives

$$\operatorname{Vol}(\Gamma_0\backslash\mathbb{H}_n)\langle f, gE_{k-\ell}\rangle_{\mathfrak{y}} = \int_{P\cap\Gamma_0\backslash\mathbb{H}_n} \prec \rho(y)f(z), g(z) \succ |y|^{s+\frac{n+1+k+\ell}{2}} d^\times z. \tag{5.3.8}$$

The same game of using the Fourier expansions of f and g, seen in Equations (2.4.7), (2.4.8), (2.4.9) and beyond, is now played out, and we obtain that the right-hand side of

(5.3.8) is equal to

$$2N(\mathfrak{x})^{-\frac{n(n+1)}{2}} \sum_{\sigma \in S_+/GL_n(\mathbb{Z})} \nu_\sigma^{-1} \prec H_{\rho,\sigma}(s) c_f(\sigma,1), c_g(\sigma,1) \succ,$$

where

$$H_{\rho,\sigma}(s) = H_{\rho,\sigma}^n\left(s; \tfrac{k+\ell}{2}\right) := \int_Y \rho(y) e^{-4\pi \operatorname{tr}(\sigma y)} |y|^{s+\frac{k+\ell}{2}} d^\times y, \tag{5.3.9}$$

and recall that

$$\nu_\sigma = \#\{a \in GL_n(\mathbb{Z}) \mid a^T \sigma a = \sigma\}.$$

Proposition 5.3.1. *Let $f \in \mathcal{S}_{\rho_k}(\Gamma, \psi)$ and $g \in \mathcal{M}_{\rho_\ell}(\Gamma', \psi')$ be as above. For a complex variable s the Rankin-Selberg Dirichlet series of f and g, defined by*

$$D_\rho(s,f,g) := \sum_{\sigma \in S_+/GL_n(\mathbb{Z})} \nu_\sigma^{-1} \prec H_{\rho,\sigma}(s) c_f(\sigma,1), c_g(\sigma,1) \succ,$$

is well defined.

Proof. If $a \in GL_n(\mathbb{Z})$ then, by Theorem 5.1.3 (iii) and (iv) along with the fact that $\eta_\infty(|a|)|a|^{[k]-[\ell]} = 1$, we have

$$\prec H_{\rho,a^T\sigma a}(s) c_g(a^T\sigma a,1), c_g(a^T\sigma a,1) \succ = \prec H_{\rho,a^T\sigma a}(s) \rho(a^T) c_f(\sigma,1), \rho(a^T) c_g(\sigma,1) \succ .$$

Then a change of variables in the definition of $H_{\rho,a^T\sigma a}$ in (5.3.9), combined with the fact that $|a^{-1} y \tilde{a}| = |y|$, gives

$$H_{\rho,a^T\sigma a}(s) = \rho(a^{-1}) H_{\rho,\sigma}(s) \rho(\tilde{a}),$$

from which we deduce the proposition. □

Hence we obtain

$$\operatorname{Vol}(\Gamma_0 \backslash \mathbb{H}_n) \langle f, g E_{k-\ell} \rangle_{\mathfrak{y}} = 2N(\mathfrak{x})^{-\frac{n(n+1)}{2}} D_\rho(s,f,g). \tag{5.3.10}$$

The above expression is the takeaway result to remember from this subsection but we finish up with an analyticity result for $D_\rho(s,f,g)$, which will be of use in Section 5.4 later to prove a non-vanishing result of the L-function.

Proposition 5.3.2. *Let f and g be as above and assume $k \geq \ell$. Then the following are true:*

(i) the series $D_\rho(s,f,g)$ can be meromorphically continued to the whole s-plane and is holomorphic for $\mathfrak{Re}(s) \geq 0$ if $k \neq \ell$ or $\mathfrak{Re}(s) > 0$ if $k = \ell$;

(ii) the sum defining $D_\rho(s,f,g)$ is absolutely convergent for $\mathfrak{Re}(s) > 0$ if g is a cusp form.

Proof.

(i) This follows from the integral expression of (5.3.10) above and the meromorphic continuation of the Eisenstein series $E_{k-\ell}(z, \bar{s} + \frac{n+1}{2})$ seen, for example, in Lemma 17.2 (4) of [Shi00].

(ii) The operator $\sqrt{H_{\rho,\sigma}} := \int_Y \rho(\sqrt{y}) e^{-2\pi\,\mathrm{tr}(\sigma y)} |y|^{\frac{2s+k+\ell}{4}} d^\times y$ is Hermitian, so that

$$D_\rho(s, f, g) = \sum_{\sigma \in S_+/GL_n(\mathbb{Z})} \nu_\sigma^{-1} \prec \sqrt{H_{\rho,\sigma}} c_f(\sigma, 1), \sqrt{H_{\rho,\sigma}} c_g(\sigma, 1) \succ .$$

Then, by the Cauchy-Schwarz inequality, we have

$$\left| \prec \sqrt{H_{\rho,\sigma}} c_f(\sigma, 1), \sqrt{H_{\rho,\sigma}} c_g(\sigma, 1) \succ \right| \leq \left[\left\{\!\!\left\{ \sqrt{H_{\rho,\sigma}} c_f(\sigma, 1) \right\}\!\!\right\} \left\{\!\!\left\{ \sqrt{H_{\rho,\sigma}} c_g(\sigma, 1) \right\}\!\!\right\} \right]^{\frac{1}{2}},$$

where $\{\!\{\cdot\}\!\}$ denotes the norm induced by the inner product $\prec \cdot, \cdot \succ$ and from which

$$|D_\rho(s, f, g)| \leq [D_\rho(s, f, f) D_\rho(s, g, g)]^{\frac{1}{2}}.$$

So the proof is finished by showing convergence of $D_\rho(s, h, h)$ when $\mathfrak{Re}(s) > 0$ and h is a cusp form. By (i) of this proposition the series $D_\rho(s, h, h)$ is holomorphic for $\mathfrak{Re}(s) > 0$, has non-negative coefficients, and hence is convergent. □

5.3.2 Dirichlet series

If f is a non-zero eigenform as in the previous section, then Lemma 4 and the subsequent discussion found in [Weis83] guarantees that there exists $\tau \in S_+$ such that $\prec c_f(\tau, 1), P(\sqrt{2\tau^{-1}}) \succ \neq 0$, where $P(x) \in \mathbb{C}[M_n]$ is the pluriharmonic polynomial that is a highest-weight vector for ρ and that forms part of the coefficients of the vector-valued theta series $\theta_{\rho,\chi}$.

Definition 5.3.3. Let $f \in \mathcal{S}_{\rho_k}(\Gamma, \psi)$ be a non-zero eigenform, let $\tau \in S_+$, and let χ be a Hecke character. Then we define the *Dirichlet series of f at τ* for a complex variable $s \in \mathbb{C}$ by

$$D_\tau^\rho(s, f, \chi) := \sum_{x \in \mathsf{X}/\mathsf{O}} (\psi\chi^*)(|x|) \prec c_f(\tau, x), P(\sqrt{2\tau^{-1}}) \succ |x|^{-s} \|x\|_{\mathbb{A}}^{-n-1}.$$

Recall the Hecke operator $\mathcal{T}_{\psi,\chi}$ defined in (2.4.11) and let $c_{\mathcal{T}}(\tau, 1) := c(\tau, 1; f|\mathcal{T}_{\psi,\chi})$. In the scalar case, one can relate $c_{\mathcal{T}}(\tau, 1)$ with the scalar Dirichlet series $D_\tau(s, f, \chi)$, this is [Shi94, Theorem 5.1] when $k \in \mathbb{Z}$ and [Shi95b, Theorem 5.1] when $k \notin \mathbb{Z}$ (which we also saw in Equation (2.4.13) of Section 2.4.3). This is done by the definition of Hecke operators and the coset decompositions of Lemma 2.2.3, all of which remain the same in the present setting, and we therefore have

$$c_{\mathcal{T}}(\tau,1) := c(\tau,1;f|\mathcal{T}_{\psi,\chi}) = \alpha_{\mathfrak{c}}(B_k\tau)\sum_{x\in \mathsf{X}/\mathsf{O}} (\psi\chi^*)(|x|)c_f(\tau,x)|x|^{-s}\|x\|_{\mathbb{A}}^{-n-1}, \tag{5.3.11}$$

where $B_k = N(\mathfrak{b}^{-1})^{2[k]-2k+1}$ and the definition of $\alpha_{\mathfrak{c}}$ depends on whether $k\in\mathbb{Z}$ or not. If $k\notin\mathbb{Z}$ then $\alpha_{\mathfrak{c}}(\zeta)$ was defined by (2.4.14) in Section 2.4.3. If $k\in\mathbb{Z}$ then $\alpha_{\mathfrak{c}}$ is given, for any $\zeta\in S_{\mathbf{f}}$ such that $e_{\mathbf{f}}(\mathrm{tr}(S_{\mathbf{f}}(\mathbb{Z})\zeta)) = 1$, by $\alpha_{\mathfrak{c}}(\zeta) := \prod_{p\nmid\mathfrak{c}}\alpha_p(\zeta_p)$ and

$$\alpha_p(\zeta_p) := \sum_{\sigma=d^{-1}c\in S_p\backslash S(\mathbb{Z}_p)} (\psi^{\mathfrak{c}}\chi^*)(|d|)e_p(-\,\mathrm{tr}(\zeta_p\sigma))|d|^{-s}; \tag{5.3.12}$$

in the above sum we have decomposed $\sigma = d^{-1}c$ into its numerator $c\in M_n(\mathbb{Z}_p)$ and denominator $d\in M_n(\mathbb{Z}_p)\cap GL_n(\mathbb{Q}_p)$, which satisfy $c\mathbb{Z}_p^n + d\mathbb{Z}_p^n = \mathbb{Z}_p^n$.

Now take $\tau\in S_+$ such that $\prec c_f(\tau,1), P(\sqrt{2\tau^{-1}})\succ\, \neq 0$. If $\Lambda(n) = \Lambda(A(n))$ are the eigenvalues of f, the identity of (2.4.12) relating $c_{\mathcal{T}}(\tau,1)$ and $c_f(\tau,1)$ holds also in this case too. So comparing $\prec c_{\mathcal{T}}(\tau,1), P(\sqrt{2\tau^{-1}})\succ$ with the identities in (2.4.12) and (5.3.11), we get

$$\left(\sum_{n=1}^{\infty}\Lambda(n)(\psi^{\mathfrak{c}}\chi^*)(n)n^{-s}\right)\prec c_f(\tau,1), P(\sqrt{2\tau^{-1}})\succ = \alpha_{\mathfrak{c}}(B_k\tau)D_\tau^\rho(s,f,\chi). \tag{5.3.13}$$

The equations of (2.4.15) remain true for vector-valued half-integral weight forms since the definition of the L-functions and abstract Hecke rings are unchanged from the scalar case. The equations of (2.4.15) also hold for vector-valued integral-weight forms when we replace $\alpha_{\mathfrak{c}}(\tau)$ with the integral-weight version $\alpha_{\mathfrak{c}}(N(\mathfrak{b}^{-1})\tau)$ defined in (5.3.12) above; this can be seen by using (5.1.4) and (5.1.5) of this thesis and (5.8) of [Shi94]. So through (2.4.15) and (5.3.13) we get

$$\prec c_f(\tau,1), P(\sqrt{2\tau^{-1}})\succ L_\psi(s,f,\chi) = \prod_{p\in\mathbf{b}} g_p\left((\psi^{\mathfrak{c}}\chi^*)(p)p^{-s}\right)\Lambda_{\mathfrak{c}}\left(\tfrac{2s-n}{4}\right)D_\tau^\rho(s,f,\chi). \tag{5.3.14}$$

This section, and everything needed to give the desired integral expression in the next section, is now completed by relating $D_\tau^\rho(s,f,\chi)$ with $D_\rho(s,f,\theta_{\rho,\chi})$.

We have $\theta_{\rho,\chi}\in\mathcal{M}_{\rho_\ell}(\Gamma',\psi')$ with $\ell = \frac{n}{2}$ and $\psi' = \chi^{-1}\epsilon_\tau$. By definition

$$c_\theta(\sigma,1) = \sum_{x\in\mathsf{X}'_\sigma}(\chi_\infty\chi^*)^{-1}(|x|)P(\sqrt{2\tau}x)$$

for $\sigma\in S_+$ and X'_σ defined by (2.4.18). Hence, replicating the calculation of (2.4.19), we get

$$D_\rho(s,f,\theta_{\rho,\chi}) = \sum_{\sigma\in S_+/GL_n(\mathbb{Z})}\nu_\sigma^{-1}\sum_{x\in\mathsf{X}'_\sigma}(\chi_\infty\chi^*)(|x|)\prec c_f(x^T\tau x,1), H_{\rho,x^T\tau x}P(\sqrt{2\tau}x)\succ, \tag{5.3.15}$$

where we used that $H_{\rho,x^T\tau x} = H_{\rho,x^T\tau x}(s)$ is Hermitian.

The following three results establish an explicit expression for $H_{\rho,x^T\tau x}(s)P(\sqrt{2\tau}x)$.

Lemma 5.3.4. *There exists an $\alpha \in \mathbb{C}^\times$ such that*

$$\int_Y P(y)e^{-4\pi\,\mathrm{tr}(y)}|y|^{s+\frac{2k+n}{4}}\,d^\times y = \alpha P(1).$$

Proof. Write $V := V_\rho$ for the representation space of ρ, and let $W := W_\varphi$ be the representation space of φ – the irreducible representation of $O(n)$ associated with ρ. Then $V \otimes W = V \otimes W^* = \mathrm{Hom}(V, W) = M_{d\times r}(\mathbb{C})$, where $d := \dim(V)$ and $r := \dim(W)$. So the group $GL_n \times O(n)$ acts on the set of $M_{d\times r}(\mathbb{C})$-valued pluriharmonic polynomials $\mathbf{P}$ on M_n by

$$\rho(g^T)\mathbf{P}(x)\,\varphi(o)^{-1} = \mathbf{P}(oxg),$$

for $o \in O_n(\mathbb{C})$ and $g \in GL_n(\mathbb{C})$. Each $\mathbf{P}$ consists of columns of V-valued polynomials P_j for $j \in \{1, \ldots, r\}$, where each P_j is pluriharmonic and $\rho(g^T)P_j(x) = P_j(xg)$. So we can choose the polynomial P to be one of the columns of some such polynomial $\mathbf{P}$. Hence the lemma is given by showing

$$\int_Y \mathbf{P}(y)e^{-4\pi\,\mathrm{tr}(y)}|y|^{s+\frac{2k+n}{4}}\,d^\times y = \alpha\mathbf{P}(1), \tag{5.3.16}$$

for some $\alpha \in \mathbb{C}^\times$ and all such $\mathbf{P}$.

The representation $\rho \otimes \varphi$ is non-trivial and $GL_n(\mathbb{C})$ is dense in $M_n(\mathbb{C})$, so that if $\mathbf{P} \not\equiv 0$ then there must exist $A \in GL_n(\mathbb{C})$ such that $\mathbf{P}(A) \neq 0$, which, by the action of ρ, tells us that $\mathbf{P}(1) \neq 0$ also. So let $U = \{\mathbf{P}(1) \mid \mathbf{P} \not\equiv 0\} \subseteq V \otimes W$, this is an invariant subspace as

$$(g, o)\cdot\mathbf{P}(1) = \rho(g)\mathbf{P}(1)\,\varphi(o) \neq 0,$$

where $\cdot$ is the action of $\rho\otimes\varphi$ on $V \otimes W$ and where the second equality follows as $\mathbf{P}(1) \neq 0$. But by assumption $\rho \otimes \varphi$ is irreducible so $U = V \otimes W$, the point of such a conclusion being that we can assume $\mathbf{P}(1) = v \otimes w$ where v is a highest-weight vector for ρ and w is a highest-weight vector for φ.

Now note that any $y \in Y$ is symmetric so it can be written as $y = a^T\delta a$, where $a \in O_n(\mathbb{C})$ and $\delta = \mathrm{diag}[\delta_1, \ldots, \delta_n]$. Then the integral of (5.3.16) becomes

$$c_0 \int_{O_n(\mathbb{R})} \rho(a^T) \int_{\mathcal{D}} \mathbf{P}(a^T\delta)e^{-4\pi\,\mathrm{tr}(\delta)}|\delta|^{s+\frac{2k+n}{4}} \left[\prod_{j<k}(\delta_k - \delta_j)\right] d^\times\delta da, \tag{5.3.17}$$

for some constant c_0, where $\mathcal{D} := \{\mathrm{diag}[\delta_1, \ldots, \delta_n] \mid \delta_i \in \mathbb{R}\}$ and $d^\times\delta = |\delta|^{-\frac{n+1}{2}}d\delta$. We have $\mathbf{P}(a^T\delta) = \delta_1^{\alpha_1}\cdots\delta_n^{\alpha_n}\mathbf{P}(a^T)$ for some $m_i \in \mathbb{Z}$, since $\mathbf{P}(a^T\delta) = \rho(\delta)\mathbf{P}(1)\,\varphi(a)$ and $\mathbf{P}(1)$ is a highest-weight vector. So the integral of (5.3.17) then becomes

$$c_0 \int_{O_n(\mathbb{R})} \rho(a^T) \int_{\mathcal{D}} \mathbf{P}(a^T)e^{-4\pi\,\mathrm{tr}(\delta)}|\delta|^{s+\frac{2k+n}{4}} \left[\prod_{i=1}^n \delta_i^{\alpha_i} \prod_{j<k}(\delta_k - \delta_j)\right] d^\times\delta da,$$

which completes the proof, since $a \in O_n(\mathbb{C})$ and therefore $\rho(a^T)\mathbf{P}(a^T) = \mathbf{P}(1)$. $\square$

Lemma 5.3.5. *The constant α of Lemma 5.3.4 is given as*

$$(4\pi)^{n(s+\frac{2k+n}{4}+\lambda_P)}\alpha = \pi^{\frac{n(n-1)}{4}}\prod_{i=0}^{n-1}\Gamma\left(s+\lambda_i+\tfrac{2k+n-2i}{4}\right),$$

where $(\lambda_1,\dots,\lambda_n)$ is the weight of $P(1)$ with respect to ρ and $\lambda_P := \lambda_1+\cdots+\lambda_n \in \mathbb{Z}$.

Proof. Notice by Lemma 5.3.4 that

$$\prec \alpha P(1), P(1) \succ = \int_Y \prec P(y), P(1) \succ e^{-4\pi\,\mathrm{tr}(y)}|y|^{s+\frac{2k+n}{4}}d^\times y. \tag{5.3.18}$$

By Gauss decomposition we can write $y = \triangledown\triangledown^T$, where $\triangledown = \triangledown(y)$ is upper triangular. Both the action of ρ on P and its behaviour with respect to $\prec\cdot,\cdot\succ$ give that $\prec P(\triangledown\triangledown^T), P(1)\succ$ $= \prec P(\triangledown), P(\triangledown)\succ$, and since $\triangledown$ is upper triangular we have $P(\triangledown) = \triangledown_{11}^{\lambda_1}\cdots\triangledown_{nn}^{\lambda_n}P(1)$. Hence the integral of (5.3.18) becomes

$$\prec P(1), P(1)\succ \int_Y e^{-4\pi\,\mathrm{tr}(y)}|y|^{s+\frac{2k+n}{4}}\left[\prod_{i=1}^{n}\triangledown(y)_{ii}^{2\lambda_i}\right]dy.$$

This latter integral has been computed by Maass in [Maa71, pp. 76–80] and is equal to

$$(4\pi)^{-n(s+\frac{2k+n}{4}+\lambda_P)}\pi^{\frac{n(n-1)}{4}}\prod_{i=0}^{n-1}\Gamma\left(s+\lambda_i+\tfrac{2k+n-2i}{4}\right).$$

□

Proposition 5.3.6. *The integral $H_{\rho,x^T\tau x}(s)P(\sqrt{2\tau}x)$ has the following expression:*

$$(4\pi)^{n(s+\frac{2k+n}{2}+\lambda_P)}H_{\rho,x^T\tau x}(s)P(\sqrt{2\tau}x) = \mathbf{\Gamma}_\rho\left(s+\tfrac{2k+n}{4}\right))P(\sqrt{2\tau^{-1}}\tilde{x}),$$

where $\lambda_P = \lambda_1+\cdots+\lambda_n$ is as in Lemma 5.3.5 and

$$\mathbf{\Gamma}_\rho(s) := \pi^{\frac{n(n-1)}{4}}\prod_{i=0}^{n-1}\Gamma\left(s+\lambda_i-\tfrac{i}{2}\right). \tag{5.3.19}$$

Proof. By definition we have

$$H_{\rho,x^T\tau x} = \rho(x^{-1}\sqrt{\tau}^{-1})H_{\rho,I_n}\rho(\sqrt{\tau}^{-1}\tilde{x})$$

so that

$$\begin{aligned}
H_{\rho,x^T\tau x}(s)P(\sqrt{2\tau}x) &= \rho(x^{-1}\sqrt{\tau}^{-1})H_{\rho,I_n}P(\sqrt{2})\\
&= \rho(x^{-1}\sqrt{2\tau^{-1}})\int_Y P(y)e^{-4\pi\,\mathrm{tr}(y)}|y|^{s+\frac{2k+n}{4}}d^\times y\\
&= (4\pi)^{-n(s+\frac{2k+n}{4}+\lambda_P)}\mathbf{\Gamma}_\rho(s)P(\sqrt{2\tau^{-1}}\tilde{x}),
\end{aligned}$$

where Lemmas 5.3.4 and 5.3.5 were used. □

In combining Proposition 5.3.6 and the expression for $D_\rho(s,f,\theta_{\rho,\chi})$ in (5.3.15), we get

$$\begin{aligned}&(4\pi)^{n\lambda_P}|4\pi\tau|^{s+\frac{2k+n}{4}}D_\rho(s,f,\theta_{\rho,\chi})\\ &\quad=\mathbf{\Gamma}_\rho\left(s+\tfrac{2k+n}{4}\right)\sum_{x\in\mathsf{X}'/\mathsf{O}'}(\chi_\infty\chi^*)(|x|)\prec c_f(x^T\tau x,1),P(\sqrt{2\tau^{-1}}\tilde{x})\succ|x|^{-2s-k-\frac{n}{2}},\end{aligned}\tag{5.3.20}$$

where, recall, $\mathsf{X}'=GL_n(\mathbb{Z})\cap GL_n(\mathbb{Q})$ and $\mathsf{O}'=GL_n(\mathbb{Z})$.

Much like [Shi94, (5.8)] if $k\in\mathbb{Z}$ and [Shi95b, (3.7)] if $k\notin\mathbb{Z}$, strong approximation tells us that

$$D^\rho_\tau(s,f,\chi)=\sum_{x\in\mathsf{X}'/\mathsf{O}'}(\psi_{\mathbf{f}}\chi^*)(|x|)\prec c_f(\tau,x),P(\sqrt{2\tau^{-1}})\succ|x|^{n+1-s}$$

which, when combined with Theorem 5.1.3 (iii), (iv), and the fact that $(\psi\chi)_\infty(|x|)=1$ for $x\in\mathsf{X}'/\mathsf{O}'$, gives

$$D^\rho_\tau(s,f,\chi)=\sum_{x\in\mathsf{X}'/\mathsf{O}'}(\chi_\infty\chi^*)(|x|)\prec c_f(x^T\tau x,1),P(\sqrt{2\tau^{-1}}\tilde{x})\succ|x|^{n+1-k-s}.\tag{5.3.21}$$

Together, the identities (5.3.20) and (5.3.21) above give

$$D^\rho_\tau(s,f,\chi)=\mathbf{\Gamma}_\rho\left(\tfrac{s-n-1+k}{4}\right)^{-1}(4\pi)^{n\lambda_P}|4\pi\tau|^{\frac{s-n-1+k}{2}}D_\rho\left(\tfrac{2s-3n-2}{4},f,\theta_{\rho,\chi}\right).\tag{5.3.22}$$

5.4 Main theorems

Notation. $k\in\frac{1}{2}\mathbb{Z}$ – integral or half-integral weight.
$[k]=k$ if $k\in\mathbb{Z}$; $[k]=k-\frac{1}{2}$ if $k\notin\mathbb{Z}$.
V – finite-dimensional complex vector space.
$\rho: GL_n(\mathbb{C})\to GL(V)$ – rational representation; $\rho_k=h^{k-[k]}\otimes\det^{[k]}\otimes\rho$.
$P(x)$ – V-valued pluriharmonic polynomial in $\mathbb{C}[M_n]$ which is a highest-weight vector for ρ and coefficient for $\theta_{\rho,\chi}$.
$\lambda_P=\lambda_1+\cdots+\lambda_n$; $(\lambda_1,\ldots,\lambda_n)$ the weight of P with respect to ρ.
$\delta=n\pmod 2\in\{0,1\}$.
$\Gamma_n(s)=\pi^{\frac{n(n-1)}{4}}\prod_{i=0}^{n-1}\Gamma(s-\frac{i}{2})$.
$\mathbf{\Gamma}_\rho(s)=\pi^{\frac{n(n-1)}{4}}\prod_{i=0}^{n-1}\Gamma(s+\lambda_i-\frac{i}{2})$.
ϵ_τ – quadratic character associated to $\mathbb{Q}(i^{[n/2]}\sqrt{|2\tau|})/\mathbb{Q}$, $\tau\in M_n(\mathbb{Q})$.
$\Lambda_{\mathfrak{a}}(s)$ – product of Dirichlet L-functions defined by (2.4.1) and (2.4.3) for an integral ideal $\mathfrak{a}$.
$g_q\in\mathbb{Z}[t]$ – polynomials satisfying $g_q(0)=1$, given by (2.4.15).

The main results of this chapter are given here; they concern the final Rankin-Selberg integral expression – a culmination of the results of the previous section – and analyticity properties of the standard L-function.

Assume that $k\geq\frac{n}{2}$ if $k-\frac{n}{2}\in\mathbb{Z}$, or $k>\frac{n}{2}$ if $k-\frac{n}{2}\notin\mathbb{Z}$. Recall the generalised Gamma function Γ_n of (2.1.13) and the notation $\delta = n \pmod 2 \in \{0,1\}$. Define some Γ-factors by

$$\Gamma^{n,k}(s) := \begin{cases} \Gamma_n\left(s + \dfrac{k-n}{2}\right) & \text{if } n < k \notin \mathbb{Z}, \\ \Gamma\left(s + \dfrac{k-n-\delta}{2} - \left[\dfrac{k-\delta}{2}\right]\right)\Gamma_n\left(s + \dfrac{k-n}{2}\right) & \text{if } n < k \in \mathbb{Z}, \\ \Gamma_{2k-n+1}\left(s + \dfrac{k-n}{2}\right)\displaystyle\prod_{i=k-\frac{n}{2}+1}^{[n/2]} \Gamma\left(2s - \dfrac{n}{2} - i\right) & \text{if } \frac{n}{2} \geq k - \frac{n}{2} \in \mathbb{Z}, \\ \Gamma_{2k-n+1}\left(s + \dfrac{k-n}{2}\right)\displaystyle\prod_{i=[k-\frac{n}{2}]+1}^{[(n-1)/2]} \Gamma\left(2s - \dfrac{n+1}{2} - i\right) & \text{if } \frac{n}{2} \geq k - \frac{n}{2} \notin \mathbb{Z}. \end{cases}$$

Theorem D1. *Let $k \in \frac{1}{2}\mathbb{Z}$ and take a fractional ideal $\mathfrak{b}$ and an integral ideal $\mathfrak{c}$ such that $(\mathfrak{b}^{-1}, \mathfrak{bc}) \subseteq 2\mathbb{Z} \times 2\mathbb{Z}$ if $k \notin \mathbb{Z}$. Let $f \in \mathcal{S}_{\rho_k}(\Gamma, \psi)$ be a non-zero eigenform, where $\Gamma = \Gamma[\mathfrak{b}^{-1}, \mathfrak{bc}]$ and ψ is a Hecke character such that $\psi_\infty(x)^n = \operatorname{sgn}(x_\infty)^{n[k]}$. Select and fix $\tau \in S_+$ such that*

$$\prec c_f(\tau, 1), P(\sqrt{2\tau^{-1}}) \succ \neq 0.$$

Let χ be another Hecke character such that $(\psi\chi)_\infty(x) = \operatorname{sgn}(x_\infty)^{[k]}$. Then

$$\begin{aligned} L_\psi(s, f, \chi)\Gamma^{n,k}\left(\tfrac{s}{2}\right) &= \left[\mathbf{\Gamma}_\rho\left(\tfrac{s-n-1+k}{2}\right) 2 \prec c_f(\tau, 1), P(\sqrt{2\tau^{-1}}) \succ\right]^{-1} \\ &\times N(\mathfrak{b})^{\frac{n(n+1)}{2}} (4\pi)^{n\lambda_P} |4\pi\tau|^{\frac{s-n-1+k}{2}} \prod_{p \in \mathbf{b}} g_p\left((\psi^{\mathfrak{c}}\chi^*)(p)p^{-s}\right) \\ &\times \left(\frac{\Lambda_{\mathfrak{c}}}{\Lambda_{\mathfrak{y}}}\right)\left(\tfrac{2s-n}{4}\right) \left\langle f, \theta_{\rho,\chi}\mathcal{E}'(\cdot, \tfrac{2s-n}{4})\right\rangle_{\mathfrak{y}} \operatorname{Vol}(\Gamma_0 \backslash \mathbb{H}_n), \end{aligned}$$

where $\mathbf{\Gamma}_\rho$ is the function of (5.3.19); $\lambda_P = \lambda_1 + \cdots + \lambda_n$, where $(\lambda_1, \ldots, \lambda_n)$ is the weight of P with respect to ρ; $g_q \in \mathbb{Z}[t]$ are polynomials, given by the relation on the second line of (2.4.15), satisfying $g_q(0) = 1$; $\Lambda_{\mathfrak{a}}(s) = \Lambda_{\mathfrak{a}}^{n,k-n/2}(s, \psi\chi\epsilon_\tau)$ is the product of Dirichlet L-functions defined by (2.4.1) for an integral ideal $\mathfrak{a}$; ϵ_τ is the quadratic character corresponding to the extension $\mathbb{Q}(i^{[\frac{n}{2}]}\sqrt{|2\tau|})/\mathbb{Q}$;

$$\mathcal{E}'(z, s) := \overline{\Gamma^{n,k}(s + \tfrac{n}{4})\Lambda_{\mathfrak{y}}(s)} E_{k-\frac{n}{2}}(z, s; \overline{\psi\chi\epsilon_\tau}, \Gamma_0);$$

$\Gamma_0 = \Gamma[\mathfrak{x}^{-1}, \mathfrak{xy}]$; and $\mathfrak{x} = \mathfrak{b} \cap \mathfrak{b}'$, $\mathfrak{y} = \mathfrak{x}^{-1}(\mathfrak{bc} \cap \mathfrak{b}'\mathfrak{c}')$, with $\mathfrak{b}'$ and $\mathfrak{c}'$ determined in (5.2.3).

Proof. The integral expression of the theorem is a combination of three auxiliary expressions obtained in the previous section:

(1) the integral expression, (5.3.10), of $D(s, f, g)$ with $g = \theta_{\rho,\chi}$;

(2) the identity, (5.3.22), relating $D(s, f, \theta_{\rho,\chi})$ with $D_\tau(s, f, \chi)$;

(3) the identity, (5.3.14), relating $D_\tau(s, f, \chi)$ with $L_\psi(s, f, \chi)$.

□

Let $Z_\psi(s, f, \chi) := \Gamma^{n,k}(\frac{s}{2})L_\psi(s, f, \chi)$, the poles of this function can be deduced using the integral expression of Theorem D1 and they come from two sources, those of the scalar-weight Eisenstein series $\mathcal{E}'(z, \frac{2s-n}{4})$ (see [Shi94, Theorem 7.3]) and those of the quotient of Dirichlet L-functions $\Lambda_{\mathfrak{c}}/\Lambda_{\mathfrak{y}}$.

Theorem D2. *Let $\frac{n}{2} \leq k \in \frac{1}{2}\mathbb{Z}$ and take a non-zero eigenform $f \in \mathcal{S}_{\rho_k}(\Gamma, \psi)$ as usual. If χ is a Hecke character such that $(\psi\chi)_\infty(x) = (x_\infty)^{[k]}$, then the function $Z_\psi(s, f, \chi)$ has only finitely many poles, all of which are simple, that are determined as follows.*

- *If $(\psi\chi)^2 \neq 1$ then $Z_\psi(s, f, \chi)$ has simple poles only for s where $\left(\frac{\Lambda_{\mathfrak{c}}}{\Lambda_{\mathfrak{y}}}\right)\left(\frac{2s-n}{4}\right)$ has poles.*
- *If $(\psi\chi)^2 = 1$ and $\mathfrak{y} \neq \mathbb{Z}$ then, in addition to the possible poles of $\left(\frac{\Lambda_{\mathfrak{c}}}{\Lambda_{\mathfrak{y}}}\right)\left(\frac{2s-n}{4}\right)$, there may be poles detailed in the following two cases.*

  (i) If $k > n$ then $Z_\psi(s, f, \chi)$ has a single pole at $s = n+1$ only if $k \in \mathbb{Z}$ and $k - n \in 2\mathbb{Z}$.

  (ii) If $\frac{n}{2} \leq k \leq n$ then the possible poles of $Z_\psi(s, f, \chi)$ occur only in the sets

$$\begin{aligned} &\{j \mid j \in \mathbb{Z}, n+1 \leq j \leq 2n+1-k\} && \text{if } k - \tfrac{n}{2} \in \mathbb{Z}, \\ &\{k + \tfrac{1}{2} \mid j \in \mathbb{Z}, n+1 \leq k \leq 2n + \tfrac{1}{2} - k\} && \text{if } k - \tfrac{n}{2} \notin \mathbb{Z}; \end{aligned}$$

  if, on the other hand, we have $(\psi\chi)^2 = 1$, $\mathfrak{y} = \mathbb{Z}$, and $k - \frac{n}{2} \in \mathbb{Z}$ then, in addition to the potential poles specified in the first set of (ii), there may also be poles in

$$\{j \in \mathbb{Z} \mid [\tfrac{n+1}{2}] \leq j \leq n\}.$$

Remark 5.4.1. The poles from the factor $\left(\frac{\Lambda_{\mathfrak{c}}}{\Lambda_{\mathfrak{y}}}\right)\left(\frac{2s-n}{4}\right)$ can be removed in two ways, either by removing the Euler factors at $p \mid \mathfrak{c}$ from the L-function and assuming $\psi = 1$, as in Theorem 2.2.6, or by imposing conditions discussed in [Shi94, Proposition 8.3].

Definition 5.4.2. Let χ_0 be an odd non-trivial character of conductor $p \neq 2$, let $\tau \in S_+$ be diagonal and $Q \in GL_n(\mathbb{Q}_{\mathbf{f}})$ be as in Lemma 5.2.5 and Proposition 5.2.6 such that, by Theorem 5.2.7, θ_{ρ,χ_0} is a cusp form. We say that $f \in \mathcal{M}_{\rho_k}(\Gamma, \psi)$ is χ_0-ordinary if both of the following are satisfied:

(i) $(\psi\chi_0)_\infty(x) = \operatorname{sgn}(x)^{[k]}$,

(ii) $\prec c_f(\tau, 1), P(\sqrt{2\tau^{-1}}) \succ \neq 0$.

Since χ_0 is odd then the condition (i) above is only possible if n is even, in which case this condition becomes $\psi_\infty(x) = \operatorname{sgn}(x_\infty)^{[k]+1}$. By Theorem 5.1.3 (iii) and (iv), the condition that τ be diagonal is non-exacting.

Theorem D3. *The function $L_\psi(s, f, \chi)$ can be meromorphically continued to the whole s-plane. Furthermore if n is even and if, for a character χ_0 of conductor $p \neq 2$, we have that f is χ_0-ordinary, then the Euler product of*

$$L_\psi^{(p)}(s, f, \chi) := L_\psi(s, f, \chi) L_p((\psi^{\mathfrak{c}}\chi^*)(p)p^{-s})$$

is convergent, and therefore non-zero, for $\mathfrak{Re}(s) > \frac{3n}{2} + 1$.

Proof. The meromorphic continuation is given by the integral expression of Theorem D1 and the continuation of the scalar-weight Eisenstein series.

Consider Equation (5.3.14) with $\chi = \chi_0$, which relates the Euler product of $L_\psi(s, f, \chi_0)$ with the Dirichlet series $D^\rho_\tau(s, f, \chi_0)$. Note that the product $\prod_{q \in \mathbf{b}} g_q((\psi^{\mathfrak{c}}\chi_0^*)(q)q^{-s})$ is just finite and since, by assumption, $\mathfrak{Re}(\frac{2s-n}{4}) \geq 1$, so is $\Lambda_{\mathfrak{y}}(\frac{2s-n}{4})$. Therefore by the formal Lemma 22.7 of [Shi00], the convergence of $L_\psi(s, f, \chi_0)$ rests on the convergence of $D^\rho_\tau(s, f, \chi_0)$ which, in turn, rests on the convergence of $D_\rho(\frac{2s-3n-2}{4}, f, \theta_{\rho,\chi_0})$ by the identity of (5.3.22). Since θ_{ρ,χ_0} is a cusp form by Theorem 5.2.7, then Proposition 5.3.2 (ii) gives that the series $D(\frac{2s-3n-2}{4}, f, \theta_{\rho,\chi_0})$ is convergent for $\mathfrak{Re}(s) > \frac{3n}{2} + 1$. Hence the convergence and non-vanishing of $L_\psi(s, f, \chi_0)$ has been established.

Now let χ be any character, and remove the Euler factor of p from $L_\psi(s, f, \chi)$ to get $L^{(p)}_\psi(s, f, \chi)$. The Euler products of both $L^{(p)}_\psi(s, f, \chi)$ and $L_\psi(s, f, \chi_0)$ range over all primes $q \neq p$, and so the Euler product of $L^{(p)}_\psi(s, f, \chi)$ is just that of $L_\psi(s, f, \chi_0)$ twisted by the $\mathbb{T}$-valued character $\chi\chi_0^{-1}$. The formal Lemma 22.7 of [Shi00] and the previous paragraph then tell us that such an Euler product is convergent and non-vanishing for $\mathfrak{Re}(s) > \frac{3n}{2} + 1$. $\square$

Chapter 6

Summation & future research

The key aim of this thesis was the capitalisation on the advanced analytic theory of metaplectic modular forms to resolve fundamental arithmetic questions relating to these forms. By turns, methods that were successfully employed in the integral-weight case have been extended to show:

(1) how the decomposition, $\mathcal{M}_k = \mathcal{S}_k \oplus \mathcal{E}_k$, of modular forms into cusp forms and Eisenstein series preserves algebraicity of the Fourier coefficients of the forms involved;

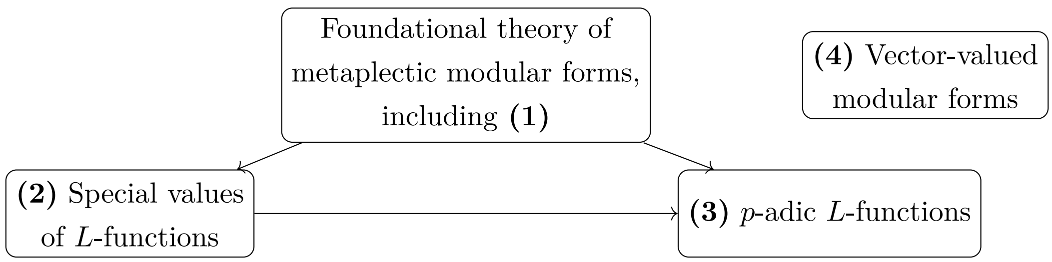
(2) precise determination of the special values of the standard L-function;

(3) the existence of p-adic L-functions interpolating the special values of the standard L-function;

(4) an explicit Rankin-Selberg integral expression for the standard L-function associated to vector-valued modular forms and resultant analytic properties of this L-function.

The results of **(2)** are a slight refinement of those by Bouganis in [Bou18]. The results of **(1)** and **(3)** are completely new and there was no indication that such results should hold *a priori*, so they are perhaps of greater significance. The Rankin-Selberg expression of **(4)** has been shown to exist previously by Piatetski-Shapiro and Rallis in [PSR88], via the language of automorphic representations, in a less explicit form with Euler factors removed. The work of this thesis is the first time an explicit Rankin-Selberg expression has been established in the vein of Shimura's of [Shi94] and [Shi96].

Listing the areas of this thesis as though they are disparate is perhaps misleading; the areas of **(1)**, **(2)**, and **(3)** are intimately related, and **(4)** is also relatively close. With arrows $A \to B$ indicating that the research area A leads naturally into the research area B, they fit together as follows:



Nevertheless, to talk of the implications of these results it is more manageable to consider each area separately. Thus, we split up the following in-depth discussion accordingly. We shall recap the results along with their limitations, their place in the literature, and future research that could follow.

The findings of the previous chapters are a consolidation of the anticipated arithmetic theory of metaplectic modular forms. It can be seen that, in spite of the struggles inherent in developing the foundational material, metaplectic modular forms behave, arithmetically, as one would like a modular form to. The similitude to the integral-weight case suggests that the metaplectic setting is not as bizarre as it first appears. Those working in the metaplectic Langlands program in developing the algebraic theory of modular forms for general m-fold metaplectic covers, and those wishing to extend Iwasawa-theoretic ideas to the present, 2-fold, metaplectic setting should find interest in this work as concrete instances of their abstract algebraic work. The establishment of analogous analytic theory and resultant arithmetic results for m-fold metaplectic modular forms – i.e. those of $m^{-1}\mathbb{Z}$ weight – would be an interesting long-term venture in this arena.

6.1 Algebraic decomposition

In Section 2.3 we showed that

$$\mathcal{M}_k(\Gamma, \psi, \mathscr{L}) = \mathcal{S}_k(\Gamma, \psi, \mathscr{L}) \oplus \mathcal{E}_k(\Gamma, \psi, \mathscr{L}),$$

where $\mathscr{L} = \mathbb{Q}(\Lambda_{k,\psi}, G(\psi), \zeta_\star)$, $\Lambda_{k,\psi}$ is the collection of all eigenvalues in $\mathcal{M}_k(\Gamma, \psi)$, $G(\psi)$ is the usual Gauss sum, and $\zeta_\star = \zeta|_X$ is the character of (2.1.1) restricted to the cusps of $\Gamma\backslash\mathbb{H}_n$. The above decomposition was shown by Shimura in this setting for $\mathscr{L} = \overline{\mathbb{Q}}$, see [Shi00], and by Harris, in [Har84], for automorphic forms associated to Shimura varieties. Fitting into neither of these two categories, our result is entirely new and was obtained by a fairly non-trivial extension of Harris' method of [Har81]. It was used as a stepping stone in this thesis to prove more special values for the L-function in Chapter 3, however the decomposition $\mathcal{M}_k = \mathcal{S}_k \oplus \mathcal{E}_k$ is fundamental in the theory of automorphic forms and therefore this result is of independent interest as well.

The key difficulty in extending the method of Harris was the need to deal with the arithmetic behaviour of f at non-trivial cusps, this is Theorem 2.3.12; we had to do this since there are no full-level metaplectic modular forms. To resolve this issue, we multiplied by the theta series of weight $\frac{1}{2}$ to obtain an integral-weight modular form and appealed to the q-expansion principle of [FC80]. Such a principle does not exist in the present setting and the inclusion of $\zeta_\star$ in $\mathscr{L}$ is a direct result of this workaround. Whether the inclusion of $\zeta_\star$ is a limitation of this particular method or a necessity is not so clear; as we discussed in Remark 2.3.13 we believe it is necessary but that one may have $\mathbb{Q}(\zeta_\star) \subseteq \mathbb{Q}(\zeta_\mathfrak{c})$ in general, which would allow the artificial removal of $\zeta_\star$ from $\mathscr{L}$. To resolve this one could either study the behaviour of ζ on the cusps or attempt to establish a q-expansion principle for this setting. The lack of algebraic theory makes the latter option very difficult at present.

In accordance with the possible future research proposed in the next subsection, one could try to extend this result to general totally real number fields. Given that the key results of Shimura that we used in this section are given over totally real number fields, we believe this should be achievable.

6.2 Algebraic L-values

In Chapter 3 we proved two theorems regarding the special values of the standard L-function. In Theorem B1 we showed that, in normalising the L-function by powers of π and other factors into $L^{\star}_{\psi}(s, f, \chi)$, we have

$$L^{\star}_{\psi}(m, f, \chi) \in \mathbb{Q}(f, \psi, \chi),$$

for a limited set of special values $m \in \Omega'_{n,k}$. In Theorem B2 we extend the set of special values to $m \in \Omega_{n,k}$ but in doing so we obtain the slightly weaker algebraicity

$$L^{\star}_{\psi}(m, f, \chi) \in \mathscr{L}(f, \chi),$$

where $\mathscr{L}$ is determined by Section 2.3 and is also given in the previous subsection.

In other settings in which one can associate algebraic motives to automorphic forms, special values of L-functions have a deeper context – see, for example, conjectures of Deligne [Del79] and Beilinson [Bei85]. The importance of such results in general is underlined by their applications to extant problems in number theory, of which the BSD conjecture is the most famous. We believe that the present results should have their place in an analogous framework and that there are therefore possibilities for fruitful applications to fundamental questions in number theory. The development of such a framework hinges on further advances in the algebraic theory of metaplectic modular forms.

The results we proved are a slight refinement of Bouganis' Theorem 6.2 in [Bou18], in that the parity condition $\mu \neq 0$ of that theorem was removed. To achieve our results we extended, in two ways, Sturm's method which was used in [Stu81] to prove special values of the standard L-function for integral-weight Siegel modular forms of even degree. The key to extending the method was to establish Sturm's explicit holomorphic projection operator in the metaplectic case. Once we did this, the extension of this method to prove Theorem B1 by using Shimura's Rankin-Selberg expression [Shi96, (4.1)] was relatively benign. Doing the same for Theorem B2 required the highly non-trivial results of Section 2.3 that we discussed in the previous subsection; we also used the ideas of Panchishkin's extension of the holomorphic projection, [Pan91], to achieve this.

The method used in proving Theorem B2 inherits the possible limitations discussed in the previous subsection in defining the field $\mathscr{L}$. The bounds on k given in Theorems B1 and B2 arise out of the definition of the constants $\mu'(\Lambda, k, \psi)$ and $\mu(\Lambda, k, \psi)$, along with the bound $k > 2n$ needed for the holomorphic projection operator. Recent work of Maurischat in [MW17] and [MW18] shows that the holomorphic projection operator produces "phantom" non-holomorphic terms for lower weights; whether this phantom projection could apply to the present setting and improve the bounds of Theorems B1 and B2 is a possible future research venture.

In [Shi00], Shimura works over general totally real number fields and shows that these L-values belong to $\overline{\mathbb{Q}}$; likewise, Theorem 6.2 of [Bou18] is also given in such generality. We believe that the methods of Chapter 3 would also extend to totally real number fields. To do so one needs, in general, to establish the following: the explicit holomorphic projection operator of Theorem 3.2.2, the bounds of Section 3.3, and the results of Section 2.3 as discussed in the previous subsection.

6.3 p-adic L-functions

In Theorem C of Chapter 4 we proved the existence of two p-adic measures $\nu_f^{\pm}$ that interpolate the special values of the standard L-function from the preceding chapter. As a result we can define two p-adic L-functions $\mathcal{L}_p^{\pm}(s, f, \chi)$ as p-adic Mellin transforms of $\nu_f^{\pm}$.

When $n = 1$, the existence of these p-adic L-functions is immediate from Shimura's correspondence and the elliptic case, nevertheless once can construct it directly and we did this in [Mer18a]. The existence of the p-adic L-function for general $n > 1$ is entirely new and is therefore more significant. The theory of p-adic L-functions is central to Iwasawa theory and, given the latter's crucial role in advances to problems such as the BSD conjecture, it is an active research area. We believe that the p-adic L-functions constructed in Chapter 4 should have an Iwasawa-theoretic interpretation and possible subsequent applications to fundamental problems in number theory; the key result of this chapter – that the analytic notion of the p-adic L-function exists – forms an important first step in this direction.

To produce the relevant p-adic measure we extended the method of Panchishkin, [Pan91], which was done for integral-weight Siegel modular forms of even degree. The key difficulty in extending this method was the definition of the p-stabilisation f_0 of an eigenform f, and to do this we used the Hecke theory developed by Shimura in [Shi95b]. The method we used was limited by the fact that it did not guarantee $f_0 \neq 0$ and so we had to make this a crucial assumption in Theorem C. In the integral-weight setting of arbitrary degree, Böcherer and Schmidt in [BS00] gave an alternative method for defining this p-stabilisation which does guarantee that $f_0 \neq 0$. It would be worth investigating whether this latter method is compatible with the methods of Chapter 4. To extend the rest of Panchishkin's methods we used Shimura's Rankin-Selberg expression of [Shi96, (4.1)] and his theory of theta series and Eisenstein series found in [Shi00]. Due to the Fourier development of the Eisenstein series $\mathcal{E}^{\star}(z, \frac{2m-n}{4})$ having support at $|\tau| = 0$ for the special value $m = n + \frac{1}{2}$ (see [Shi00, Proposition 17.6]), we excluded this value entirely from the interpolation (and likewise for the value $m = n + \frac{3}{2}$ in certain cases). This was largely for convenience and one should be able to use the explicit Fourier coefficients for τ with rank $< n$, of [Shi96, Proposition 16.10], and the Mazur measure to amend the formula of Theorem C and interpolate this value accordingly.

A distant future research proposal following on from this chapter is in the development of the Iwasawa-theoretic notion of the p-adic L-function and the Iwasawa main conjecture in this setting. The algebraic tools needed for this are not currently available, though progress is being made with respect to the metaplectic Langlands program.

6.4 Vector-valued modular forms

Chapter 5 consisted of joint work with Bouganis and concerned V-valued modular forms of weight $\rho_k = h^{k-[k]} \otimes \det^{[k]} \otimes \rho$, where $\rho : GL_n(\mathbb{C}) \to GL(V)$ is an irreducible representation of a vector space V. In Theorem D1 we gave a very explicit Rankin-Selberg integral expression relating the standard L-function $L_\psi(s, f, \chi)$ of a vector-valued eigenform f with the integral

$$\left\langle f, \theta_{\rho,\chi}\mathcal{E}'(\cdot, \tfrac{2s-n}{4})\right\rangle.$$

As immediate consequences of this we proved two analyticity results concerning the L-function: Theorem D2 gave the location of all possible poles of $L_\psi(s, f, \chi)$, and Theorem D3 gave a partial non-vanishing criterion for the L-function with Euler factors removed.

Much of the prior work in computing L-values for vector-valued modular forms has made use of the doubling method – see [Tak92] and [Koz00]. Being the first explicit Rankin-Selberg expression established in the style of Shimura, our expression is significant in that it paves the way for alternative proofs for these L-values and it is often the case that these two methods compute different sets of values.

The method of Shimura in [Shi94], which was applied to scalar Siegel modular forms of integral weight, was extended to the vector-valued setting. To do this we had to define three key objects: a particular vector-valued theta series whose Fourier coefficients were highest-weight vectors with respect to ρ (Section 5.2), the appropriate Rankin-Selberg Dirichlet series $D_\rho(s, f, g)$ of two vector-valued modular forms f and g (Definition 5.3.1), and the analogous Dirichlet series $D^\rho_\tau(s, f, \chi)$ that associates to $L_\psi(s, f, \chi)$ (Definition 5.3.3). The condition on the Fourier coefficients of the theta series was needed in order to calculate the integral $H_{\rho,\sigma}$ of (5.3.9) which, under this condition, becomes Γ-factors and allowed us to relate the two Dirichlet series $D_\rho(s, f, \theta_{\rho,\chi})$ and $D^\rho_\tau(s, f, \chi)$ (see (5.3.22)). The method was thus ultimately limited by the set of representations ρ we could consider, those for which there exists a pluriharmonic polynomial acting as a highest-weight vector, and by the key fact that $\rho \otimes \det$ is not one. In the scalar case, given a character χ, one chose $\mu \in \{0, 1\}$ depending on the parity of χ and then used the theta series of weight $\det^\mu \otimes \det^{\frac{n}{2}}$ in the Rankin-Selberg method; we were unable to analogously take $(\rho \otimes \det^\mu) \otimes \det^{\frac{n}{2}}$ in Chapter 5. Therefore the expression of Theorem D1 only holds for characters χ of certain parity, in particular for $(\psi\chi)_\infty(-1) = (-1)^{[k]}$, as do the resultant analyticity results. That we could not take $\mu = 1$ had even further implications for the non-vanishing result of Theorem D3. The method of that theorem hinges on the theta series being a cusp form and in the scalar case one could use that $\theta_\chi^{(1)}(z;\tau)$ is a cusp form, see [Shi00, (A3.16)]. So we needed to attack the difficult issue of cuspidality of theta series of weight $\rho_{\frac{n}{2}}$, this was Theorem 5.2.7. It required the further assumption that χ be odd and of prime conductor $p \neq 2$, which latter meant the removal of the Euler factor at p from Theorem D3. The odd parity of χ is only compatible with the condition $(\psi\chi)_\infty(-1) = (-1)^{[k]}$ if n is even and, moreover in that case, this limits the parity of ψ to be $\psi_\infty(-1) = (-1)^{[k]+1}$. So the non-vanishing result is heavily limited and can only be considered a partial result.

The original aim of the work of Chapter 5 was to prove $\overline{\mathbb{Q}}$-algebraicity of L-values by the method of [Shi00, Theorem 28.8]. For such a result to be effective relative to the doubling method, a full result on non-vanishing for $\mathfrak{Re}(s) > \frac{3n}{2} + 1$ is required, which we were not able to provide. Proving such a non-vanishing result, and subsequent algebraicity, would be a worthwhile future research problem. This could be possible by trying to calculate the integral $H_{\rho,\sigma}$ through alternative means or by proving a stronger cuspidality result on the vector-valued theta series of Section 5.2.

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